Morita Equivalence and Generalized Kähler Geometry

by

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Abstract

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Generalized Kähler (GK) geometry is a generalization of Kähler geometry, which arises in the study of supersymmetric sigma models in physics. In this thesis, we solve the problem of determining the underlying degrees of freedom for the class of GK structures of symplectic type. This is achieved by giving a reformulation of the geometry whereby it is represented by a pair of holomorphic Poisson structures, a holomorphic symplectic Morita equivalence relating them, and a Lagrangian brane inside of the Morita equivalence.

We apply this reformulation to solve the longstanding problem of representing the metric of a GK structure in terms of a real-valued potential function. This generalizes the situation in Kähler geometry, where the metric can be expressed in terms of the partial derivatives of a function. This result relies on the fact that the metric of a GK structure corresponds to a Lagrangian brane, which can be represented via the method of generating functions. We then apply this result to give new constructions of GK structures, including examples on toric surfaces.

Next, we study the Picard group of a holomorphic Poisson structure, and explore its relationship to GK geometry. We then apply our results to the deformation theory of GK structures, and explain how a GK metric can be deformed by flowing the Lagrangian brane along a Hamiltonian vector field. Finally, we prove a normal form result, which says that locally, a GK structure of symplectic type is determined by a holomorphic Poisson structure and a time-dependent real-valued function, via a Hamiltonian flow construction.

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Contents

Chapter 1

Introduction

1.1 History and Motivation

This thesis is about a new approach to the field of *generalized Kähler (GK) geometry*: using tools from the theory of Poisson geometry, and the theory of Lie groupoids, we give a reformulation of GK geometry which clarifies the nature of the underlying geometric structures, and allows us to solve a longstanding problem in the field.

This subject has its origins in the work of physicists of the 1980s, who were then studying *supersymmetric non-linear sigma models.* The basic idea is very simple: we start with a 'space-time' Σ (in our case, a surface), equipped with a metric h, and consider mapping it into a Riemannian manifold (M, g). These give the 'fields' of the theory. Then, to each field, that is to say, to each map $\phi : \Sigma \to M$, we attach a number, the 'action', which very roughly has an expression of the form

$$S(\phi) = \int_\Sigma ||d\phi||^2, \tag{1.1}$$

where $||d\phi||^2$ uses the metrics h and g to measure the norm of $d\phi$. This theory of maps is what is known as the *non-linear sigma model.* This theory has for symmetries the isometries of (Σ, h), but physicists often try look for even more symmetries. A theorem of Coleman and Mandula [23] puts a strong limitation on where to look: in order to find interesting new symmetries extending the usual Lie algebra of isometries, it is in fact necessary to consider *super* Lie algebras [52], which means that the elements carry a degree grading, according to which they are either odd or even. This is the idea of supersymmetry: the original algebra of isometries consists of even elements, and to these we add the odd generators of supersymmetry. The sigma model in our story is the one with so-called $N = (2, 2)$ supersymmetry; this is where the connection with geometry emerges.

In 1979, Zumino [117] discovered that one source of supersymmetry for the sigma model comes from the geometry of the target manifold. Specifically, he found that, after enlarging the field theory by adding in new fields, the resulting model has $N = (2, 2)$ supersymmetry if (M, g) is equipped with a *Kähler* structure. Recall that a Kähler structure on (M, g) consists of a complex structure I, which is compatible with the metric g, in the sense that g is Hermitian, and such that the corresponding differential 2-form $\omega = gI$ is closed. The payoff of the extra symmetry is immediate: the action for this theory can be written in terms of a single (locally defined) real function K on M. Because this theory reduces to the one above, the metric g involved in the action 1.1 is a consequence of the function K. Precisely, the components of g are given by

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j},$$

where z_i are complex coordinates associated to the complex structure I. Such a function K is well-known to geometers; it is called the *Kähler potential*. Whereas a generic Riemannian metric g on an n dimensional manifold involves $\frac{1}{2}n(n+1)$ independent component functions, a Kähler metric reduces to the data of a single function. This represents an enormous simplification in the geometry. For example, it is used in an essential way when looking for metrics with prescribed curvature properties, or in the study of the Kähler-Ricci flow, because it allows the problems to be reduced to differential equations involving a single function. The Kähler potential is also fundamental to our understanding of the independent 'moving parts' involved in a Kähler structure: the moduli of Kähler geometry consists of the underlying complex manifold, and the data of a real function determining the metric.

In 1984, Gates, Hull, and Roček [37] discovered that there are more general structures on the target which give rise to supersymmetry. They showed that for models with a B-field, the model admits $N = (2,2)$ supersymmetry if the target (M, g) is equipped with a *generalized Kähler* structure. This consists of a pair of complex structures I_+ and I_- on M, such that g is Hermitian with respect to both complex structures, and such that the corresponding differential 2-forms $\omega_\pm = gI_\pm$ satisfy the following integrability conditions:

$$d^c_+\omega_+ + d^c_-\omega_- = 0, \qquad dd^c_\pm\omega_\pm = 0,$$

where $d^c_\pm = i(\bar{\partial}_\pm - \partial_\pm)$ are real operators determined by the complex structures. This is the geometry which forms the focus of this thesis.

There is a natural question which immediately arises: does the metric g in a GK structure also reduce to a single real function, as in the Kähler case? Such a function would be called a *generalized Kähler potential*. From the outset, there has been much evidence in favour of this: in their original work, Gates, Hull and Roček were able to formulate the sigma model, in the case that I_+ and I_- commute, so that the action again involves a single real-valued function on M. Much later, Lindström, Roček, von Unge, and Zabzine [71, 73] were able to extend this reformulation to the case where the commutator $[I_+, I_-]$ has constant rank. As in the Kähler case, such a formulation of the sigma model implies a description of the metric in terms of the derivatives of a single function. A general account of the GK potential, however, has remained out of reach.

A more fundamental question concerns the underlying degrees of freedom inherent in GK geometry. We would like a description which clearly identifies what is the holomorphic geometry underlying a GK structure, and what extra data is needed to specify the metric. A major obstruction to such a description is the fact that we have 2 distinct complex structures I_+ and I_- on the same smooth manifold. We need a description of the geometry that sees both these complex structures as emerging from a single underlying holomorphic geometry.

Progress on understanding generalized Kähler geometry remained limited until the early 2000s. But in 2003, Hitchin [55] introduced the field of generalized complex geometry, and Gualtieri [44] showed that GK geometry admits a reformulation in terms of a pair of commuting generalized complex structures. This development led to a renewed interest in the geometry, and a number of new advances were made. Most notably for our story, Hitchin [56] made the surprising discovery that underlying a generalized

Kähler manifold are two holomorphic Poisson structures. More precisely, what he observed was that the tensor

$$Q = [I_-, I_+]g^{-1} : T^*M \to TM,$$

is the common imaginary part of two holomorphic Poisson structures $\sigma_\pm$, with respect to $I_\pm$ respectively. This observation gives an important clue to the question of determining the underlying holomorphic structure of GK geometry, and it forms the starting point for this thesis.

In this thesis, we study the class of generalized Kähler structures of *symplectic type*, where $I_+ + I_-$ is assumed to be invertible, and solve the problem of determining the underlying degrees of freedom. This is done by giving a reformulation of the geometry, whereby it is represented as a Lagrangian brane inside of a holomorphic symplectic Morita equivalence, which relates the two holomorphic Poisson structures described by Hitchin. This identifies the Morita equivalence as the underlying holomorphic structure, and the Lagrangian brane as the extra data needed to describe the metric. As an application of this approach, we solve the longstanding problem of describing the metric in terms of a single real valued function, thereby providing a generalized Kähler potential even in the case where the commutator $[I_+, I_-]$ changes rank.

The main technical tool required by our approach is the notion of a Morita equivalence between Poisson manifolds, which was developed by the Weinstein school of Poisson geometry [111, 108]. That the theories of Poisson geometry, and Lie groupoids, are required in order to properly understand GK geometry, is quite unexpected from the perspective of the original definition given by the physicists. From the vantage point of this thesis, however, it becomes natural, and we can see in a precise way how generalized Kähler geometry is a non-linear generalization of Kähler geometry. Much of this thesis is devoted to exploring the ways in which the new groupoid theoretic approach can be fruitfully applied. In particular, we explore the applications to the deformation theory of GK metrics, and give several new examples of GK structures.

1.2 A taste of what's to come

Before going further, let's take a second look at ordinary Kähler geometry, and explore a reformulation that was first considered by Donaldson in his study of the complex Monge-Ampère equation [33]. As it turns out, this is the approach that must be generalized to GK geometry.

Let (M, I, g) be a Kähler manifold. The idea is to use the associated Kähler form $\omega = gI$ to deform the holomorphic cotangent bundle T^*M. Since ω is a real closed $(1,1)$ form, it can locally be expressed in terms of a real valued potential function K as follows: $\omega = i\partial\bar{\partial}K$. This is the Kähler potential. Now, choose a cover of the manifold M by open sets U_i on which potential functions K_i have been chosen. On each double overlap $U_i \cap U_j$, the $(1,0)$ form $\mu_{ij} = i\partial(K_i - K_j)$ is closed and holomorphic, and so we can define the following holomorphic affine transformation:

$$A_{ij} : T^*U_i|_{U_i \cap U_j} \to T^*U_j|_{U_i \cap U_j}, \qquad \alpha_x \mapsto \alpha_x + \mu_{ij}(x).$$

In fact, because μ_{ij} is closed, A_{ij} also preserves the canonical symplectic form Ω_0 on the cotangent bundle:

$$A_{ij}^*(\Omega_0) = \Omega_0 + \pi^*\mu_{ij}^*(\Omega_0) = \Omega_0 + \pi^* d\mu_{ij} = \Omega_0,$$

where $\pi : T^*M \to M$ is the vector bundle projection. Incidentally, μ_{ij} gives a Čech representative of the Kähler class:

$$[\omega] = [\mu_{ij}] \in H^1((M, I), \Omega^1).$$

But for our present purpose, we interpret this class as an affine deformation of the cotangent bundle. Indeed, the A_{ij} define a cocycle, which can be used to re-glue the pieces T^*U_i into a manifold Z, in the following way:

$$Z = (\coprod_i T^*U_i)/\sim, \quad (\alpha_x)_i \sim (A_{ij}(\alpha_x))_j.$$

The space Z is a holomorphic affine bundle modelled on the holomorphic cotangent bundle T^*M, and it is equipped with a holomorphic symplectic structure Ω. In fact, something more is true: the symplectic structure on Z is compatible with the additive action of T^*M, in the sense that the graph of the addition map

$$A : T^*M \times_M Z \to Z, \qquad (\alpha_x, \beta_x) \mapsto \alpha_x + \beta_x,$$

is a Lagrangian submanifold of the product $(T^*M, \Omega_0) \times (Z, \Omega) \times (Z, -\Omega)$. The space Z is our first example of a Morita equivalence, and we can see from its construction that it encodes both the holomorphic structure on M, and the cohomology class of ω.

To see where the metric g comes in, we need to take another look at the potential functions. If we define $\mathcal{L}_i = -\partial K_i : U_i \to T^*U_i$, then on double overlaps

$$A_{ij} \circ \mathcal{L}_i(x) = -i\partial K_i + \mu_{ij} = -i\partial K_i + i\partial(K_i - K_j) = \mathcal{L}_j,$$

and so we get a global section $\mathcal{L} : M \to Z$. This section fails to be holomorphic, but if we use it to pullback the form Ω, then we see from a local calculation that we get

$$\mathcal{L}^*(\Omega) = \mathcal{L}_i^*(\Omega_0) = d(-i\partial K_i) = i\partial\overline{\partial} K_i = \omega.$$

Therefore, the section $\mathcal{L}$ allows us to recover the Kähler form, and hence also the metric g. Submanifolds such as $\mathcal{L}$ are known as *brane bisections*; these are defined to be sections of the projection $Z \to M$, which are *Lagrangian* with respect to the imaginary part $\mathrm{Im}\Omega$. Brane bisections give a global meaning to the Kähler potentials, and they are the extra data that allow us to encode the metric.

There is one more observation that we can make about the affine bundle Z. Using the section $\mathcal{L}$ and the addition map, we get a diffeomorphism between the underlying smooth manifolds

$$\psi : T^*M \to Z, \qquad \alpha_x \mapsto \alpha_x + \mathcal{L}(x).$$

Then, using the compatibility between the symplectic structures and the addition map, we see that when we pullback the form Ω, we get

$$\psi^*\Omega = \Omega_0 + \pi^*\mathcal{L}^*\Omega = \Omega_0 + \pi^*\omega.$$

Hence, the affine bundle (Z, Ω) is nothing but the *twisted cotangent bundle* $(T^*M, \Omega_0 + \pi^*\omega)$, with the brane bisection given by the zero section.

Using this model for the affine bundle, we can now study what happens when we deform $\mathcal{L}$ in a fixed (Z, Ω). Let $\mathcal{L}'$ be another brane bisection. Viewing it as a section of T^*M, it is given by the graph of a

$(1,0)$-form α. Pulling back the symplectic form, we get

$$\omega' := (\mathcal{L}')^*\Omega = \alpha^*(\Omega_0 + \pi^*\omega) = \omega + d\alpha.$$

The condition that $\mathcal{L}'$ is a brane means that ω' is real, and using the $\partial\bar{\partial}$-lemma, we conclude that $\omega' - \omega = i\partial\bar{\partial}f$, for a real-valued function f. Hence, varying $\mathcal{L}$ in a fixed (Z,Ω) corresponds to varying ω within its Kähler class.

Proposition 1.2.1. *[33, Section 2.] On a Kähler manifold, the cohomology class $[\omega]$ of the Kähler form determines a holomorphic symplectic affine bundle (Z,Ω) modelled on the cotangent bundle, and the metric g determines a smooth section $\mathcal{L}$ of the bundle Z which is symplectic for* $\mathrm{Re}(\Omega)$ *and Lagrangian for* $\mathrm{Im}(\Omega)$*. Conversely, this data $(Z,\Omega,\mathcal{L})$ uniquely determines the Kähler structure. Under this correspondence, deforming $\mathcal{L}$ is equivalent to varying ω within the Kähler class.*

What emerges from this discussion is that the underlying holomorphic geometry of a Kähler structure corresponds to the Morita equivalence (Z,Ω), and the extra data needed to specify the metric corresponds to the brane bisection $\mathcal{L}$. This is the picture that extends to generalized Kähler structures of symplectic type. Note that a Kähler structure (M,I,g) gives a special example of a GK structure, where $I_+ = I_- = I$. In this case, the Hitchin Poisson structure vanishes:

$$Q = [I,I]g^{-1} = 0,$$

and so we see that (Z,Ω) is a Morita self-equivalence of the zero Poisson structure on (M,I). A general GK structure will also be represented by a Lagrangian brane inside of a holomorphic symplectic manifold, and as we will see, this manifold is intimately related to the Hitchin Poisson structure.

1.3 Outline of the thesis

This thesis is organized as follows. Chapter 2 gives an introduction to the theory of Lie groupoids and Poisson geometry, focussing on the important notion of Morita equivalence. This material is well-known and many illustrative examples are discussed. Section 2.1 is on Lie groupoids, and covers many of the basic features, including Lie algebroids, and bisections. Section 2.1.2 is about Morita equivalences, and provides a short account of the use of Lie groupoids in the theory of differentiable stacks. Section 2.2 is about Poisson geometry, in both the smooth and holomorphic settings. This section discusses the important definitions of symplectic groupoids, and symplectic Morita equivalences. Furthermore, several detailed examples are worked out.

In Chapter 3, we give an introduction to Generalized Kähler (GK) geometry, focussing on GK geometry of symplectic type. This material is also well-known, and all of the proofs are included. In an effort to reduce prerequisites, the presentation avoids the use of Dirac geometry. The chapter starts with a discussion of GK geometry from the bihermitian perspective, and moves on to GK geometry of symplectic type in Section 3.1. The Hitchin Poisson structure is discussed in Section 3.1.1, degenerate GK structures of symplectic type are discussed in Section 3.1.2, and the interpretation in terms of GC geometry is discussed in Section 3.1.3.

Chapter 4 presents the first main results of this thesis, describing a new approach to generalized Kähler geometry of symplectic type, which makes the underlying degrees of freedom explicit. In Section

4.1, we describe the work of [4], allowing us to associate a holomorphic symplectic Morita equivalence to a degenerate GK structure of symplectic type, which is then interpreted as the underlying holomorphic geometry. In Section 4.2, we define brane bisections, and then prove Theorem 4.2.2, which gives an equivalence between degenerate GK structures of symplectic type, and holomorphic symplectic Morita equivalences equipped with a brane bisection. We then explain in Section 4.2.1 how a metric arises from the geometry of a brane in a Morita equivalence. It follows from this that the brane bisection corresponds to the choice of a metric. A few basic examples are then presented in Section 4.3. In Section 4.4, we define the notion of prequantization of a GK structure, and prove a quantization condition in Theorem 4.4.5. Finally, in Section 4.5, we provide a Čech description of Morita equivalences with brane bisections, generalizing the description of Kähler geometry from Section 1.2.

In Chapter 5, we solve the problem of describing a GK metric in terms of a real potential function (Theorem 5.0.1). A brane bisection is, in particular, a Lagrangian submanifold of a real symplectic manifold, and therefore it can be described using a generating function once a Darboux coordinate chart has been chosen. We call this function the *generalized Kähler potential*, because of the correspondence between brane bisections and GK metrics. The rest of the chapter is devoted to examples. We first check that we recover the usual Kähler potential, as well as the known example from [114] where the Hitchin Poisson structure is non-degenerate. Next, we use the generalized Kähler potential to construct new examples of GK structures, where the Hitchin Poisson structure changes rank. In particular, Proposition 5.1.5 gives an example of a GK structure on $\mathbb{R}^4$ where the metric is complete, and Example 5.1.8 gives a family of GK structures parametrized by $0 \leq t \leq 1$, which is Kähler at $t = 0$, and complete for $t < 1$. Finally, in Section 5.2 we use a Čech description to produce GK structures on toric surfaces. Theorem 5.2.3, which is similar to Corollary 1 in [10], shows that any toric Kähler surface deforms into a family of degenarate toric GK surfaces for all $t \in \mathbb{R}$, and gives an explicit bound for t, such that the metric is positive-definite.

Chapter 6 is about the *Picard groupoid*, which is the category of holomorphic Poisson structures and holomorphic symplectic Morita equivalences. This category is closely related to generalized Kähler geometry. Indeed, GK structures of symplectic type may be interpreted as morphisms in an upgraded version of this category. Real versions of this category have been studied in [15, 13], and in this chapter we prove analogues of their results in the holomorphic setting. In Section 6.1, we investigate the infinitesimal analogues, or Picard algebras, and construct the corresponding exponential maps.

In Chapter 7, we discuss two applications of our approach to GK geometry. First, in Section 7.1, we explain how elements in the Picard group of a Poisson manifold can be used to deform generalized Kähler structures. As a special instance of this construction, we show in Proposition 7.1.1 that GK structures can be deformed by flowing the brane bisection via a Hamiltonian vector field. This construction specializes to the deformation constructions of [2, 57, 46]. In Section 7.2, we prove Theorem 7.2.2, which states that locally, any GK structure of symplectic type is obtained by applying the Hamiltonian flow of a time dependent function to a brane in a canonical degenerate GK structure associated to a holomorphic Poisson manifold. This gives an alternate generalization of the Kähler potential, and shows that a GK structure of symplectic type is locally determined by the data of a holomorphic Poisson structure and a time-dependent real-valued function.

1.4 Context and literature

Generalized Kähler geometry, in the form of bihermitian geometry, was first discovered by Gates, Hull, and Roček [37], following the work of Zumino [117] and Alvarez-Gaume and Freedman [1], which studied the relationship between supersymmetric sigma models and the geometry of the target manifold. It was rediscovered and studied from the perspective of twistor theory in [94, 87, 64, 2]. A significant development was the introduction of Generalized Complex geometry by Hitchin [55], and the reformulation of Generalized Kähler geometry by Gualtieri [44, 48].

There are several important questions in the study of Generalized Kähler geometry, many of which go back to its inception. This includes the problem of generalizing known results and constructions from Kähler geometry, such as Hodge theory [45, 19, 21, 20], generalized Kähler blow-ups [22, 102], and generalized Kähler reduction [12, 70, 107].

Another important problem is the search for examples. In addition to the examples provided by the methods just cited, important work in this direction includes [56], which uses a symmetry argument, and [57, 46], which take advantage of the underlying Poisson geometry. An important source of examples comes from deformations, due to the stability theorem of Goto [39, 38, 40].

A significant direction of research is the study of special metrics and their connection to algebraic geometry. The Calabi-Yau equation was introduced in [44] and studied in [3], generalizations of the scalar and Ricci curvature were studied in [10, 41], the Kobayashi-Hitchin correspondence was studied in [59, 42, 43], and the generalized Kähler-Ricci flow was introduced in [100, 101], and further studied in [97, 98, 99].

Finally, the problem of describing a generalized Kähler metric in terms of a potential function has existed since the work of [37], which already deals with the case of commuting complex structures. Further progress in this direction is contained in [71, 72, 73], and the relationship to gerbes is studied in [61, 114].

This thesis contributes to the field of generalized Kähler geometry by developing a new reformulation of the geometry. This allows us to address the problem of the generalized Kähler potential, covering the new case where the Hitchin Poisson structure changes rank. It contributes to the search for examples, providing a new method of construction. It contributes to the deformation theory, elucidating and generalizing the constructions of [2, 57, 46]. And it contributes to our understanding of the local structure of the geometry, via the normal form theorem (Theorem 7.2.2). The new perspective on GK geometry provided by this thesis may also prove useful in some of the other areas of research outlined above, such as the study of special metrics, where a description of the metric in terms of a single function is sought after.

1.5 Overlap with previous work

This thesis has significant overlap with [6], which was written during the course of my PhD, and represents joint work with M. Gualtieri and M. Zabzine. All of the text included in this thesis has been written by myself, but some of it is copied directly from [6]. The precise areas of overlap are the following. Section 1.2 is copied almost directly. There is some overlap in Chapter 2, such as in the examples of Section 2.2, and Definition 2.2.21 and Theorem 2.2.23 are directly copied. Some of the material of Chapter 3 coincides with Section 1 of [6], but it is presented from a different perspective. Lemma 3.1.4 and

Proposition 3.1.5 come from Lemma 1.6, and Theorem 3.1.10 corresponds to Theorem 1.5. In Chapter 4, Sections 4.2 and 4.3 are mostly copied directly from Section 4 of [6]. The beginning of Chapter 5, and the material up to, but not including, Example 5.1.6, is copied from Section 5 of [6]. And most of the material from Chapters 6 and 7 is copied directly from Sections 6, 7, and 8 of [6], although it has been somewhat rearranged and expanded.

Chapter 2

Preliminaries

In this chapter, we recall the basic theory of Lie groupoids and Poisson geometry that will be required in the thesis. Standard references for the theory of Lie groupoids are [76, 28, 79], and for the theory of Poisson geometry are [34, 67, 36]. These theories have traditionally been developed in the smooth category, but they carry over to the holomorphic category without much modification. A reference for holomorphic Poisson geometry is the paper [86].

2.1 Lie Groupoids

A concise, but unenlightening definition of a groupoid is a category where all morphisms are invertible. More precisely, a groupoid consists of a *space of objects* B, and a *space of arrows* $\mathcal{G}$ going between the objects. Hence, an arrow $g \in \mathcal{G}$ goes between its *source* $s(g) \in B$ and its *target* $t(g) \in B$, giving rise to the source and target maps $s, t : \mathcal{G} \to B$. We think of an element $g \in \mathcal{G}$ as in the following picture:

$$\begin{tikzcd} t(g) & s(g) \arrow[l, bend right, "g"'] \end{tikzcd}$$

Given two arrows g and h, such that $s(g) = t(h)$, there is a composition, or multiplication, denoted $m(g,h) = gh$, and this defines a map $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)}$ is the collection of composable arrows

$$\mathcal{G}^{(2)} = \{(g,h) \in \mathcal{G} \times \mathcal{G} | s(g) = t(h)\}.$$

This composition is required to be associative, in the sense that $(gh)k = g(hk)$, whenever this is well-defined. Next, for each object $x \in B$, there is an identity arrow $\epsilon_x \in \mathcal{G}$, going from x to x, with the property that for any $g \in \mathcal{G}$, we have $\epsilon_{t(g)} g = g = g\epsilon_{s(g)}$, and this gives rise to a map $\epsilon : B \to \mathcal{G}$, the *identity bisection*. We often identify B with its image in $\mathcal{G}$. Finally, there is an inverse map $\iota : \mathcal{G} \to \mathcal{G}$, sending each arrow to its inverse. We will usually suppress notation and denote a groupoid by $\mathcal{G} \rightrightarrows B$, or simply $\mathcal{G}$, when the context is clear.

A Lie groupoid is a groupoid internal to the category of smooth manifolds.

Definition 2.1.1 (Lie groupoid)**.** A *Lie groupoid* is a groupoid $(\mathcal{G}, B, s, t, m, \epsilon, \iota)$, where $\mathcal{G}$ and B are smooth manifolds, all maps are smooth, and s and t are surjective submersions.

A morphism of Lie groupoids from $\mathcal{G}$ to $\mathcal{H}$ is a smooth map $\Phi : \mathcal{G} \to \mathcal{H}$ that intertwines all the structure maps.

Remark 2.1.2. The manifold of arrows $\mathcal{G}$ may sometimes be non-Hausdorff, while still satisfying all other axioms of a smooth manifold. However, the base and source fibres are always assumed to be Hausdorff.

There are two important special cases of the definition. First, a groupoid for which the base B is a point is nothing but a Lie group. Second, a groupoid for which the space of arrows $\mathcal{G}$ contains only identities (i.e. $B \cong \mathcal{G}$) is nothing but a smooth manifold. Therefore, both the categories of smooth manifolds, and of Lie groups embed into the category **Gpd** of Lie groupoids. There are a few more simple examples that we can consider.

Example 2.1.3. Given a manifold X, the pair groupoid $Pair(X) \rightrightarrows X$, is the Lie groupoid over X whose space of arrows is given by the product $X \times X$, with source and target given by right and left projections, respectively, and with multiplication given by $m((a, b), (b, c)) = (a, c)$.

This example has the following generalization. Given a submersion $f : X \to Y$, the submersion groupoid $Sub(f) \rightrightarrows X$ is the subgroupoid of $Pair(X)$ whose space of arrows is given by the fibre product $X \times_Y X$. The pair groupoid is given by the submersion groupoid of the map from X to a point.
■

Example 2.1.4. Given a manifold X, the *fundamental groupoid* $\Pi(X) \rightrightarrows X$ is the Lie groupoid over X whose arrows consist of paths in X modulo homotopies fixing the endpoints. The source and target of a path γ are, respectively, the start and end points $\gamma(0)$ and $\gamma(1)$, and multiplication is given by concatenation of paths. ■

Example 2.1.5. A vector bundle $\pi : E \to X$ gives rise to a groupoid, where $s = t = \pi$, and where the groupoid multiplication is given by fibre-wise addition. ■

Example 2.1.6. Given a manifold X with a smooth right action of a Lie group G

$$\theta : X \times G \to X,$$

we can form the associated *action groupoid* $X \rtimes G \rightrightarrows X$, whose space of arrows is given by the product $X \times G$, with target given by left projection, and source given by θ. The multiplication map is given by the group product: $m((x, g)(xg, h)) = (x, gh)$. There is a similar definition of the action groupoid associated to a left action. ■

Orbits and Isotropy

A groupoid $\mathcal{G} \rightrightarrows B$ determines an equivalence relation on the space of objects. Namely, for $x, y \in B$, we have that $x \sim y$ if and only if there is some $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$. Thinking of a groupoid as a category, this is just the relation of isomorphism. The equivalence classes for this relation are called the *orbits* of the groupoid. The orbit through x, denoted $\mathcal{G}x$, is given by $t(s^{-1}(x))$, and is an immersed submanifold of B.

The *isotropy group* of $\mathcal{G}$, at a point $x \in B$, is the group $\mathcal{G}(x) = t^{-1}(x) \cap s^{-1}(x)$, consisting of all arrows going to and from the object x. It is an embedded submanifold of the space of arrows, and furthermore, $t|_{s^{-1}(x)} : s^{-1}(x) \to \mathcal{G}x$ is a principal $\mathcal{G}(x)$-bundle over the orbit.

The isotropy groups over the points of a fixed orbit are all isomorphic, although not in a canonical way. To see this, choose two points x and y in a single orbit, and let $g \in \mathcal{G}$ be such that $t(g) = x$, and

$s(g) = y$. Then conjugation by the element g establishes an isomorphism between $\mathcal{G}(x)$ and $\mathcal{G}(y)$.

Example 2.1.7. The pair and fundamental groupoids of a space X are *transitive*, meaning that they have a single orbit. The orbits of the submersion groupoid of $f : X \to Y$, are given by the fibres of f. The orbits of the groupoid associated to a vector bundle are the points of the base X. Finally, the orbits of the action groupoid $X \rtimes G$ are simply the orbits of the group action.

It is also easy to compute the isotropy groups of our examples. For the pair and submersion groupoids, they are all trivial. For the fundamental groupoid of X, the isotropy group at x coincides with the fundamental group $\pi_1(X, x)$. For the groupoid associated to a vector bundle $E \to X$, the isotropy groups are given by the individual fibres. And finally, the isotropy groups of the action groupoid are given by the stabilizer groups of the action. ■

Bisections

An important notion in the theory of Lie groupoids is that of *bisections*.

Definition 2.1.8 (Bisection)**.** A *bisection* of a groupoid $\mathcal{G}$ is a submanifold $S \subseteq \mathcal{G}$, such that both $s|_S$ and $t|_S$ are diffeomorphisms onto the base.

An equivalent way of expressing a bisection is as a morphism $\lambda : B \to \mathcal{G}$, which is a section of the source, meaning that $s \circ \lambda = id_B$, and such that $t \circ \lambda$ is a diffeomorphism. The two formulations are related by the identity $S = \mathrm{Im}(\lambda)$. Of course, it is also possible to formulate the definition in terms of sections of the target, and in this thesis we will switch between all 3 formulations according to what is most convenient.

Using the composition of arrows, we can multiply two bisections:

$$\lambda_1 * \lambda_2(x) = \lambda_1(t \circ \lambda_2(x))\lambda_2(x).$$

Under this product, the bisections form a group, denoted $Bis(\mathcal{G})$, with the identity element given by the identity bisection ϵ. Note that $Bis(\mathcal{G})$ is often an infinite dimensional Lie group.

Example 2.1.9. The group of bisections of the pair groupoid $Pair(X)$ is isomorphic to the diffeomorphism group of X. In order to see this, observe that a bisection of the pair groupoid is a map

$$\lambda : X \to X \times X, \qquad x \to (f(x), g(x)),$$

such that $g(x) = s \circ \lambda(x) = x$, and such that $f(x) = t \circ \lambda(x)$ is a diffeomorphism. Given bisections $\lambda_1(x) = (f_1(x), x)$ and $\lambda_2(x) = (f_2(x), x)$, their product is given by $(f_1 \circ f_2(x), x)$, and so we see that the product of bisections corresponds to the composition of diffeomorphisms.

The bisections of the submersion groupoid $Sub(f)$ correspond to the diffeomorphisms ϕ of X which commute with f, meaning that $f \circ \phi = f$. ■

Example 2.1.10. The group of bisections of a vector bundle $E \to X$, viewed as a groupoid, is given by the space of global sections $\Gamma(E)$. ■

The group of bisections $Bis(\mathcal{G})$ acts on $\mathcal{G}$ in several ways. There is a natural left action given by left multiplication:

$$L_\lambda : \mathcal{G} \to \mathcal{G}, \qquad g \mapsto \lambda(t(g))g.$$

Similarly, there is a right action R on $\mathcal{G}$ given by multiplication on the right, and the two can be combined to give a conjugation action $C_\lambda = L_\lambda \circ (R_\lambda)^{-1}$, which is here given as a left action. Of all these actions, it is only conjugation which acts via groupoid morphisms.

It is often also useful to consider *local bisections*, which are submanifolds $S \subseteq \mathcal{G}$ with the property that $s|_S$ and $t|_S$ are diffeomorphisms onto open subsets of the base. Alternatively, a local bisection is a map $\lambda : U \to \mathcal{G}$, defined on an open subset $U \subseteq B$, such that $s \circ \lambda = id_U$, and such that $t \circ \lambda$ is a diffeomorphism from U onto another open subset of B.

Groupoid Actions

Just as groups can act on manifolds, there is also a notion of action by Lie groupoids. Let $\mathcal{G} \rightrightarrows B$ be a Lie groupoid, and let X be a manifold. In order for $\mathcal{G}$ to act on X, we need first of all to relate X to the base B of the groupoid, which is to say that we need a *moment map* $\phi : X \to B$. An element $g \in \mathcal{G}$, which is an arrow from $s(g)$ to $t(g)$, can then act on X by mapping $\phi^{-1}(s(g))$ to $\phi^{-1}(t(g))$. In other words, an action of $\mathcal{G}$ on X is a way of lifting the arrows of $\mathcal{G}$ to the map $\phi : X \to B$. We can make this precise as follows.

Definition 2.1.11 (Groupoid action). A *left action* of $\mathcal{G}$ on X is given by a map $\phi : X \to B$ (the *moment map*), and a map

$$\theta : G^s \times^{\phi}_B X \to X, \qquad (g, x) \mapsto g.x,$$

such that $\phi(g.x) = t(g)$, $\epsilon_{\phi(x)}.x = x$, and $g.(g'.x) = (gg').x$, whenever these are well-defined, for $g, g' \in \mathcal{G}$ and $x \in X$. A *right action* is defined similarly.

If $\mathcal{G}$ is a group, then we recover the notion of a group action. At the other extreme, if $\mathcal{G}$ corresponds to a smooth manifold, so that $\mathcal{G} = B = M$, then a $\mathcal{G}$ action on X is simply given by a map $\phi : X \to M$. Every groupoid acts on its base in a tautological way: the moment map is the identity, and the action map is given by the target $t : \mathcal{G} = \mathcal{G}^s \times^{id}_B B \to B$. The orbits for this action are precisely the orbits of the groupoid.

Let us consider some more examples.

Example 2.1.12. A left action of the pair groupoid $Pair(X)$ on a space M is given by the data of a map $\phi : M \to X$, and a map $\theta : X \times M \to M$, written as $\theta(x, m) = x.m$, such that $\phi(x.m) = x$, $\phi(m).m = m$, and $y.(x.m) = y.m$. This description simplifies considerably. Namely, the space M can always be taken to have the form $X \times N$, for some manifold N, with ϕ given by projection onto the first factor, and with the action given by $y.(x, n) = (y, n)$. We can see this as follows.

First, for every point $m \in M$, there is a section of ϕ which passes through this point. Namely, let $\theta^{(m)} : X \to M$, be defined by $\theta^{(m)}(x) = x.m$. Then $\phi(\theta^{(m)}(x)) = \phi(x.m) = x$, and $\theta^{(m)}(\phi(m)) = \phi(m).m = m$. Hence ϕ is a surjective submersion. Next, choose some $x \in X$, and let $N = \phi^{-1}(x)$, which we know to be a smooth embedded submanifold. We claim that the restricted map $\theta : X \times N \to M$ is a diffeomorphism. Indeed, the inverse is simply given by sending $m \in M$ to $(\phi(m), x.m)$. It is then routine to check that the action has the above specified form. ■

Example 2.1.13. Let $X \rtimes G$ be the action groupoid of a right G-action on X. Then a right action of this groupoid on a space M is given by a map $\phi : M \to X$, and a right action of G on M, such that the map ϕ is G-equivariant. ■

As a special case of a groupoid action, we have the notion of a *groupoid representation*: in this case, the moment map is given by a vector bundle over the base, $\phi : E \to B$, and the groupoid acts by linear transformations. If we apply this to the example of the action groupoid $X \rtimes G$, then we see that its representations are given by G-equivariant vector bundles over X.

Finally, we can consider the notion of groupoid bundles and principal bundles.

Definition 2.1.14 (Groupoid bundle)**.** A left *$\mathcal{G}$-bundle* consists of a morphism $\pi : P \to M$, and a left action of $\mathcal{G}$ on P, such that the fibres of π are preserved by the action. The bundle is *principal* if π is a surjective submersion, and the action is *free* and *fiberwise transitive*, meaning that the following map is an isomorphism:

$$\mathcal{G} \times_B P \to P \times_M P, \quad (g, p) \mapsto (g.p, p)$$

There is a similar definition of a right (principal) $\mathcal{G}$-bundle

A canonical example of a left principal $\mathcal{G}$-bundle is given by the source map $s : \mathcal{G} \to B$, where the $\mathcal{G}$ action is given by multiplication, with moment map given by the target t. Similarly, $t : \mathcal{G} \to B$ is a right principal $\mathcal{G}$-bundle, with moment map given by s.

2.1.1 Lie Algebroids and Integration

We have seen that Lie groupoids give a generalization of the theory of Lie groups. Therefore, it is natural to ask about their infinitesimal counterparts, and whether there is a corresponding generalization of the theory of Lie algebras. This is indeed the case, as was first observed by Pradines [88].

Definition 2.1.15 (Lie algebroid)**.** A *Lie algebroid* consists of a vector bundle $\mathcal{A} \to B$, a bracket $[\ ,\ ] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ on its sheaf of sections, and a vector bundle morphism $\rho : \mathcal{A} \to TB$, called the *anchor* map, such that the anchor preserves brackets, and such that the following *Liebniz rule* is satisfied:

$$[a, fb] = f[a, b] + \rho(a)(f)b,$$

where a, b are sections of $\mathcal{A}$, and f is a function.

Note that the definition reduces to that of a Lie algebra when the base B is a point. A Lie algebroid can be thought of as a replacement for the tangent bundle, and indeed its sections act as derivations of smooth functions via the anchor map.

Lie algebroids are the infinitesimal counterparts of Lie groupoids, and there is a Lie functor, which associates a Lie algebroid in a canonical way to every Lie groupoid.

Proposition 2.1.16. *[88] A Lie groupoid $\mathcal{G} \rightrightarrows B$ determines a canonical Lie algebroid Lie$(\mathcal{G})$. The underlying vector bundle is given by*

$$ker(ds)|_B \subseteq T\mathcal{G}|_B,$$

the anchor is given by the restriction of dt, and the bracket is given by the Lie bracket of vector fields on $\mathcal{G}$, restricted to the right-invariant vector fields which are tangent to $ker(ds)$.

Remark 2.1.17. Note that in this proposition, we make use of the concept of *right-invariant vector fields.* A vector field X on $\mathcal{G}$, which is tangent to the source fibres, is said to be right-invariant, if it is invariant under the right action of the group of bisections. Because each element of the groupoid is related to the identity bisection by multiplication by a unique element (i.e. itself), the right invariant vector fields are in bijection with the sections of the Lie algebroid. It is this fact that allows us to define the Lie bracket.

Let us revisit the examples above.

Example 2.1.18. The Lie algebroid of a Lie group G is simply the Lie algebra of G. Given a manifold M, viewed as a Lie groupoid, its Lie algebroid is given by the 0 vector bundle over M. ■

Example 2.1.19. Both $Pair(X)$ and $\Pi(X)$ have the same Lie algebroid, which is equal to the tangent bundle TX. The Lie algebroid of the submersion groupoid $Sub(f)$ consists of the vector fields which are tangent to the fibres of $f : X \to Y$. More generally, any involutive distribution $D \subseteq TX$ on X gives an example of a Lie algebroid. ■

Example 2.1.20. Consider a vector bundle $E \to X$, viewed as a Lie groupoid. Then its Lie algebroid is given by E itself, with zero anchor map, and zero bracket. ■

Example 2.1.21. Consider again the action groupoid $X \rtimes G$ associated to the right action of G on X. Differentiating this action gives rise to a Lie algebra action $\rho : \mathfrak{g} \to \mathfrak{X}(X)$. The Lie algebroid of $X \rtimes G$ is given by the *action algebroid* $X \rtimes \mathfrak{g}$ of ρ. This is given by the trivial bundle $X \times \mathfrak{g}$, with anchor given by the action map ρ, and with bracket obtained by using the Liebniz rule to extend the bracket on $\mathfrak{g}$, viewed as the space of constant sections. ■

A generalization of Lie's second theorem holds for Lie groupoids.

Proposition 2.1.22. *[75, 81] Let $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ be Lie groupoids, such that the fibres of the source map of $\mathcal{G}$ are connected and simply-connected, and let $\phi : Lie(\mathcal{G}) \to Lie(\mathcal{H})$ be a morphism of their Lie algebroids over $f : X \to Y$. Then there exists a unique morphism of Lie groupoids $\Phi : \mathcal{G} \to \mathcal{H}$ over f such that $Lie(\Phi) = \phi$.*

Let us define an *integration* of a Lie algebroid $\mathcal{A}$ to be any Lie groupoid $\mathcal{G}$, such that $Lie(\mathcal{G}) = \mathcal{A}$. The generalization of Lie's third theorem, namely whether every Lie algebroid admits an integration, fails in general. A Lie algebroid which admits an integration is called *integrable*. The problem of determining whether a given Lie algebroid is integrable is quite subtle, but it was solved by Crainic and Fernandes [26], inspired by the work of Weinstein and of Cattaneo and Felder [18]. Partial results in this direction were also given by Mackenzie [74], Debord [31], and Nistor [84]. In the holomorphic category, Laurent-Gengoux, Stiénon and Xu [68] proved that a holomorphic Lie algebroid is integrable to a holomorphic Lie groupoid if, and only if, its underlying real (or imaginary) Lie algebroid is integrable in the smooth category. In this thesis, we will always assume that our Lie algebroids are integrable.

As we have seen in the example of the tangent bundle, an integrable Lie algebroid may admit several different integrations. However, there is always a unique integration with the property that the fibres of its source map are both connected and simply connected [79, Proposition 6.8]. This is the so-called *source simply connected (ssc)* integration.

A sketch of integration

We will outline a very rough sketch of a construction of the source simply connected integration of a Lie algebroid, following [26]. This construction was suggested by Weinstein, inspired by the work of Duistermaat and Kolk [35]. A similar construction also appeared in the work of Cattaneo and Felder [18]. Let $\mathcal{A}$ be an integrable Lie algebroid, with source simply connected integration $\mathcal{G} \rightrightarrows B$. Since the source fibres of $\mathcal{G}$ are connected and simply connected, for every arrow $g \in \mathcal{G}$, there is a path in the source fibre connecting $x = s(g)$ to g, unique up to homotopies in $s^{-1}(x)$. So let $P(\mathcal{G}) = \{g : [0,1] \to \mathcal{G} \mid g(0) = \epsilon(x),\ s(g(t)) = x\}$, and let $\sim$ denote the equivalence through homotopies restricted to lie in a single source fibre. Then $\mathcal{G} \cong P(\mathcal{G})/\sim$. Let us now relate this to the algebroid $\mathcal{A}$.

First, observe that a path $g(t) \in P(\mathcal{G})$ maps to the path $\gamma(t) = t \circ g(t)$, which lies in a single orbit of the groupoid. By itself this is not equivalent to g, but if we include the data of the derivative $g'(t)$, then we can recover g. More precisely, the speed of g at a given time, $g'(t) \in \ker(ds)_{g(t)}$, can be right translated to the identity bisection to give an element of the Lie algebroid

$$a(t) = g'(t)(g(t))^{-1} \in \mathcal{A}_{\gamma(t)}.$$

Therefore, we get a map $a : [0,1] \to \mathcal{A}$, which lies above $\gamma(t)$, and satisfies $\rho(a(t)) = \gamma'(t)$. Maps such as a are called *$\mathcal{A}$-paths*, and we denote the collection of all such paths $P(\mathcal{A})$. Then, there is a bijection between $P(\mathcal{G})$ and $P(\mathcal{A})$. Note that the condition of being an $\mathcal{A}$-path is consistent with the fact that the tangent space to an orbit of the groupoid is given by the image of the anchor map.

It is also possible to translate the above notion of homotopy for groupoid paths to a definition of $\mathcal{A}$-homotopy between $\mathcal{A}$-paths, which is defined purely in terms of the Lie algebroid. Then, we have the following bijection

$$P(\mathcal{A})/\sim\ \cong\ P(\mathcal{G})/\sim\ \cong\ \mathcal{G}.$$

But now note that we can define $P(\mathcal{A})/\sim$, even without having an integrating groupoid, since it is defined purely in terms of the algebroid. Therefore, this gives a method for constructing $\mathcal{G}$. A priori, it only defines a topological groupoid. The obstructions to endowing this space with a smooth structure, making it into a Lie groupoid, were determined in [26].

It is very useful at this point to revisit the basic example of the tangent algebroid TM. In this case there is a single orbit, and TM-paths and TM-homotopies coincide with the usual definitions of paths and homotopies. Therefore, in this example, we recover the construction of the fundamental groupoid $\Pi(M)$. We can view the above construction as a vast generalization of the fundamental groupoid construction to setting of Lie algebroids.

Bisections

The sections of a Lie algebroid $\Gamma(\mathcal{A})$ form an (often infinite-dimensional) Lie algebra. Given a groupoid $\mathcal{G}$, one should think of $\Gamma(Lie(\mathcal{G}))$ as the Lie algebra of the group of bisections $Bis(\mathcal{G})$. Indeed, consider a family λ_t of bisections, starting at the identity. At every point $b \in B$, $\lambda_t(b)$ is a path in the source fibre starting at $\epsilon(b)$, and therefore $\frac{d}{dt}|_0(\lambda_t) \in \Gamma(Lie(\mathcal{G}))$.

There is also an exponential map $\exp : \Gamma(Lie(\mathcal{G})) \to Bis(\mathcal{G})$, but it is only partially defined. It is given as follows. Starting with a section $a \in \Gamma(Lie(\mathcal{G}))$, we convert it to a right-invariant vector field a^R on the groupoid, which is tangent to the source fibres, and t-related to $\rho(a)$. We then take the time-t flow (if it exists), ϕ_t^a, and apply it to the identity bisection, giving rise to the family of bisections $\exp(ta) = \phi_t^a \circ \epsilon$. The exponential map is then defined by setting $t = 1$. Note that $t \circ \exp(ta)$ is given by the time-t flow of the vector field $\rho(a)$.

Lemma 2.1.23. *The flow ϕ_t^a of a^R is given by left-multiplication by the bisection* $\exp(ta)$.

Proof. Consider the left multiplication action $L_{\exp(ta)}$. Given an arrow g, with target $x = t(g)$, we get a curve

$$L_{\exp(t\alpha)}(g) = \phi^a_t(x)g = R_g(\phi^a_t(x)),$$

whose derivative is given by

$$\frac{d}{dt}R_g(\phi^a_t(x)) = dR_g(a^R(\phi^a_t(x))) = a^R(\phi^a_t(x)g) = a^R(L_{\exp(t\alpha)}(g)).$$

Then, since both $L_{\exp(t\alpha)}$ and ϕ^a_t agree at $t = 0$, we conclude that they are equal. □

As an immediate corollary, we see that $\exp(t_1 a) * \exp(t_2 a) = \exp((t_1+t_2)a)$, and hence the exponential map defines a 1-parameter subgroup.

2.1.2 Morita equivalence

A fundamental notion in the theory of Lie groupoids is that of Morita equivalence. This defines an equivalence relation between Lie groupoids that is weaker than isomorphism, and the study of Lie groupoids, taken up to Morita equivalence, gives rise to the theory of differentiable stacks.

One way of introducing the idea goes as follows. A Lie groupoid is a way of formalizing the notion of symmetry on a space, and as we have seen, every Lie groupoid acts on its own base in a tautological way. The notion of Morita equivalence arises when we try to study the geometry of B which is invariant under this action. Roughly, we want to do geometry on the space of orbits, $B/\mathcal{G}$, but in a more sophisticated way which remembers the isotropy groups. Thinking of Lie groupoids as categories suggests an obvious way to implement this idea. Namely, two groupoids represent the same invariant geometry if they are equivalent as categories, which we take to mean that there is a fully faithful, essentially surjective functor relating them (see [79], section 5.4, for a precise definition). This is the notion of Morita equivalence. However, as it stands, this fails to define an equivalence relation because it is not symmetric: unlike the case for ordinary categories, such functors between Lie groupoids do not necessarily have quasi-inverses. Therefore, we need to invert these functors by hand. There are different ways of doing this, and the approach to Morita equivalence taken in this thesis is based on the use of principal bibundles. This approach, and the bicategories of groupoids which result, were originally introduced in [54, 53, 89]. See also, [11, 80], where the ideas were introduced independently in the context of topos theory. Other important references are [82, 83].

Definition 2.1.24 (Morita equivalence)**.** A *Morita equivalence* between groupoids $\mathcal{G} \rightrightarrows B$ and $\mathcal{H} \rightrightarrows C$ consists of a span

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ B & & C \end{tikzcd}$$

such that

1. $q : P \to C$ is a left principal $\mathcal{G}$ bundle with moment map p;

2. $p : P \to B$ is a right principal $\mathcal{H}$ bundle with moment map q;

3. the actions of $\mathcal{G}$ and $\mathcal{H}$ commute, in the sense that $(g.p).h = g.(p.h)$, whenever this is well-defined.

In this case we say that the groupoids $\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent*.

Example 2.1.25. As we have seen, $\mathcal{G}$ is both a left and right principal $\mathcal{G}$-bundle. Since the left and right multiplication actions commute by associativity of the product, $\mathcal{G}$ therefore defines a Morita self-equivalence of itself. ■

In this thesis, we will mainly be interested in doing geometry on the space P of a Morita equivalence itself, and will not concern ourselves much with the interpretation in terms of the geometry of the orbit space. However, since it is sometimes useful to think in this way, we end this section by sketching some of the ideas involved in thinking of groupoids up to Morita equivalence as quotient stacks.

Example 2.1.26. An illuminating example to consider is that of a free group action which has a well-defined quotient. So let $\pi : P \to B$ be a right principal G-bundle, and consider the action groupoid $P \rtimes G$. As we have seen, the orbits of this groupoid coincide with the orbits of the group action, and so the orbit space is given by $P/G = B$. Therefore, we expect $P \rtimes G$ to be Morita equivalent to B. Indeed, the Morita equivalence is given by P itself:

$$\begin{tikzcd} & P \arrow[dl, "id"'] \arrow[dr, "\pi"] & \\ P & & B, \end{tikzcd}$$

where we use the fact that a B action on P is simply given by a map, such as π, and a $P \rtimes G$ action is inherited from the G action. ■

We can really see the advantage of using Lie groupoids to represent quotients in the case where the action is not free, even if the quotient is a well-defined smooth manifold. For example, given the trivial action of a group G on a space X, the 'naive' quotient $X/G = X$ is a smooth manifold, but the data of the stabilizer groups, namely G, is completely lost. On the other hand, the Morita equivalence class of $X \rtimes G$ retains this data. This is a general fact about Morita equivalences.

Proposition 2.1.27. *A Morita equivalence P between $\mathcal{G} \rightrightarrows B$ and $\mathcal{H} \rightrightarrows C$ induces a bijection between the orbit spaces, and the isotropy groups over corresponding orbits are isomorphic.*

Proof. Using the properties of principal bundles, we obtain the following set-theoretic bijection between orbit spaces

$$B/\mathcal{G} \to C/\mathcal{H}, \qquad \mathcal{G}b \mapsto q(p^{-1}(\mathcal{G}b)).$$

Next, choose two points $b \in B$ and $c \in C$, whose orbits are related by this map. Then $P(b,c) = p^{-1}(b) \cap q^{-1}(c)$ is an embedded submanifold of P, and defines a Morita equivalence between the isotropy groups $\mathcal{G}(b)$ and $\mathcal{H}(c)$. Therefore, choosing some $x \in P(b,c)$, we get a group isomorphism $\phi_x : \mathcal{G}(b) \to \mathcal{H}(c)$, defined by the property that $x.\phi_x(g) = g.x$. □

Remark 2.1.28. See Theorem 4.3.1 of [32] for a complete description, along these lines, of the Morita invariant geometry of a Lie groupoid. In particular, it is shown that the above map of orbit spaces is a homeomorphism.

Example 2.1.29. Recall that both $Pair(X)$ and $\Pi(X)$ are transitive, meaning that they have a single orbit. For such groupoids, the only Morita invariant data is their isotropy groups. Therefore, $Pair(X)$ is Morita equivalent to a point (compare this fact to example 2.1.12), and $\Pi(X)$ is Morita equivalent to the fundamental group of X. In a similar vein, the submersion groupoid of $f : X \to Y$ is Morita equivalent to Y. ■

One of the upshots of this discussion is that the Morita equivalence class of a groupoid encodes the geometry of the orbit space. Therefore, structures on a groupoid which are invariant under Morita equivalence should be thought of as representing structures on the quotient space. For example, one can show that Morita equivalent groupoids have equivalent categories of representations; therefore, representations of a groupoid correspond to vector bundles on the quotient. In the case that the quotient is a smooth manifold, we can verify that this is actually the case.

Example 2.1.30. Consider again a right principal G-bundle $\pi : P \to B$. We have seen in Example 2.1.26 that the action groupoid $P \rtimes G$ is Morita equivalent to B. Representations of the action groupoid are given by G-equivariant vector bundles over P. These are vector bundles which carry a linear G-action, for which the bundle projection is equivariant. By taking G-equivariant sections of such a bundle, we can produce a vector bundle over B. Conversely, the pullback to P of any vector bundle on B carries a natural equivariant G-structure. This gives a correspondence between the representations of $P \rtimes G$, and the vector bundles over B. ■

A more sophisticated way of thinking about quotient spaces is via the theory of *differentiable stacks*. And it turns out that Lie groupoids, taken up to Morita equivalence, give a model for differentiable stacks. To make this precise requires a few more definitions. See [5, 8, 77, 66] for a more detailed treatment of the following. First, we can extend the notion of Morita equivalence, and define a new category **dStack** of 'stacks', whose objects are Lie groupoids, and whose morphisms are given by *Morita maps*. A Morita map P from $\mathcal{H}$ to $\mathcal{G}$ is defined to be a $\mathcal{G} - \mathcal{H}$ bibundle, but with the only requirement being that the $\mathcal{G}$ action is principal. If P is a map from $\mathcal{H} \rightrightarrows C$ to $\mathcal{G} \rightrightarrows B$, and R is a map from $\mathcal{G} \rightrightarrows B$ to $\mathcal{K} \rightrightarrows A$, then their composition is given by $(R \times_B P)/\mathcal{G}$. In fact, this composition is not strictly associative, and so **dStack** is really a bicategory, with 2-morphisms given by maps between bibundles which intertwine all the structure. We can obtain a genuine category by taking isomorphism classes of Morita maps. The equivalences in **dStack** are precisely the Morita equivalences. Therefore, whenever we view a groupoid $\mathcal{G}$ as an object of **dStack**, we use the notation $[B/\mathcal{G}]$, in order to indicate that this stack represents the orbit space quotient.

The category **dStack** can be embedded into a category of stacks over the site of smooth manifolds and open coverings. Very roughly, a groupoid $\mathcal{G}$ is sent to the category of principal $\mathcal{G}$-bundles, which is fibered over the category of smooth manifolds. The image of this functor coincides with the category of differentiable stacks, as it is traditionally defined. Hence, working with groupoids up to Morita equivalence gives a concrete way of thinking about differentiable stacks, which avoids the abstract formalism of fibered categories.

Note that there is a functor from the category of groupoids **Gpd** to the category **dStack**. Restricting this functor gives an embedding of the category of smooth manifolds into **dStack**. However, this functor is not full in general, as there are often many more Morita morphisms than groupoid morphisms.

Example 2.1.31. Let X be a smooth manifold, and G a Lie group. Viewed in the category of Lie groupoids, there is a unique morphism from X to G, given by sending all points of X to the unit in G. On the other hand, the Morita maps from X to G are given by G-principal bundles over X. This example shows that even though we can embed manifolds and Lie groups into the same category of Lie groupoids, there aren't any interesting morphisms going between them. On the other hand, passing to the bicategory **dStack** allows spaces and groups to interact in a rich way. ■

The following example shows that the functor from **Gpd** to **dStack** also fails to be faithful.

Example 2.1.32. Let G and H be Lie groups, and consider a Morita map P from H to G. This is a left G-torsor, which also carries a right action of H. Choosing a point $p \in P$ defines a homomorphism $\phi_p : H \to G$, via the equality $\phi(h).p = p.h$. Note that this only works to define a map from H to G, because solving the equation $g.p = q$ relies on the fact that P is a G-torsor. The homomorphism depends on the choice of the point p up to conjugation in G, in the sense that $\phi_{g.p} = g\phi_p g^{-1}$. Since an automorphism of the Morita map P has the effect of changing the choice of the reference point p, we conclude that the isomorphism classes of Morita maps are given by the conjugacy classes of homomorphisms from H to G. ■

We end with a few more comments about **dStack**. The identity morphism of the object $[B/\mathcal{G}]$ in **dStack** is given by the Morita equivalence $\mathcal{G}$. Furthermore, given a groupoid $\mathcal{G} \rightrightarrows B$, there is a canonical Morita map $B \to [B/\mathcal{G}]$, given by $\mathcal{G}$. Such a map is called a *presentation* of the stack, and allows us to recover the groupoid as a fibre product: $\mathcal{G} = B \times_{[B/\mathcal{G}]} B$. Note that it is the manifold of arrows that is given by the fibre product, rather than the stack represented by the groupoid. In fact, it is possible to recover the groupoid multiplication, and all other structure maps, from the presentation of the stack. If $\mathcal{H} \rightrightarrows C$ is another groupoid, such that there is an isomorphism between the quotient stacks $[B/\mathcal{G}] \cong [C/\mathcal{H}]$, then the fibre product $B \times_{[B/\mathcal{G}]} C$ gives the total space of a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$.

Example 2.1.33. We can combine the previous two examples. Given a manifold X, and a Lie group G, the isomorphism classes of maps from $[x/\pi_1(X,x)]$ to $[pt/G]$ are the conjugacy classes of homomorphisms $\pi_1(X,x) \to G$. Given any such map, we can pull it back along $X \to [X/\pi_1(X)] \cong [x/\pi_1(X,x)]$ to get a map from X to $[pt/G]$, which is a principal G-bundle over X. In fact, a homomorphism $\pi_1(X,x) \to G$ corresponds to a flat G-bundle over X, and by pulling back along $X \to [X/\pi_1(X)]$, we are simply extracting the underlying bundle. ■

Bisections

It makes sense to consider bisections of a Morita equivalence P. These are submanifolds $S \subseteq P$ such that the two projections p and q restrict to isomorphisms. Morita equivalences with bisections compose: given $S_1 \subseteq R$, and $S_2 \subseteq P$, one simply takes the image of $S_1 \times_B S_2$ in the quotient $(R \times_B P)/\mathcal{G}$.

Note however, that a Morita equivalence with a bisection is in some sense trivial. Given a Morita equivalence P between $\mathcal{G}$ and $\mathcal{H}$, equipped with a bisection λ, which is here given by a section of q, we construct an isomorphism of groupoids $\Phi : \mathcal{H} \to \mathcal{G}$, covering the map $\phi = p \circ \lambda : C \to B$. It is given by the following formula

$$\Phi(h).\lambda(s(h)) = \lambda(t(h)).h.$$

The Morita equivalence is then isomorphic to the trivial Morita equivalence $\mathcal{H}$, via the map

$$T : \mathcal{H} \to P, \qquad h \mapsto \lambda(t(h)).h,$$

which sends the identity bisection to S, and satisfies $q \circ T = s$, and $p \circ T = \phi \circ t$, where s and t are, respectively, the source and target maps of $\mathcal{H}$. The right action of $\mathcal{H}$ is given by right multiplication, and the left action of $\mathcal{G}$ is given by mapping an element into $\mathcal{H}$ via Φ^{-1}, followed by left multiplication.

2.2 Poisson Geometry

Definition 2.2.1 (Poisson bracket)**.** A *Poisson bracket* on a manifold M is a Lie bracket on the smooth functions

$$\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

which satisfies the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

The pair $(M, \{-,-\})$ is called a Poisson manifold.

A morphism of Poisson manifolds $\phi : (M, \{-,-\}_M) \to (N, \{-,-\}_N)$ is a smooth map $\phi : M \to N$, such that for $f, g \in C^\infty(N)$, we have $\phi^*\{f, g\}_N = \{\phi^* f, \phi^* g\}_M$.

Note that the Liebniz rule implies that the operator $\{f, -\} : C^\infty(M) \to C^\infty(M)$ is a derivation, and hence corresponds to a vector field $X_f \in \mathfrak{X}(M)$, which is called the *Hamiltonian* vector field of f. Skew-symmetry of the bracket implies that it is a derivation in each argument. Therefore, a Poisson bracket corresponds to a *bivector field* $Q \in \Gamma(\wedge^2 TM)$, which can be defined by the formula

$$Q(df, dg) = \{f, g\},$$

for smooth functions $f, g \in C^\infty(M)$. In fact, this formula allows us to define a skew-symmetric bracket satisfying the Leibniz rule for any bivector field. The Jacobi identity for the bracket then corresponds to the following identity

$$[Q, Q] = 0,$$

where $[-,-]$ is the *Schouten bracket*, a graded Lie bracket on multi-vector fields extending the Lie bracket of vector fields. See section 1.8 of [34] for a definition of the Schouten bracket. The definition of a Poisson morphism can also be stated in terms of bivector fields. Namely, given Poisson manifolds, (M, Q_M) and (N, Q_N), a morphism $\phi : M \to N$ is Poisson if $\phi_*(Q_M) = Q_N$. Such a morphism is called *anti-Poisson* if instead $\phi_*(Q_M) = -Q_N$.

Using the bivector Q, we can define a map of vector bundles via interior product

$$Q : T^*M \to TM, \qquad \alpha \mapsto \iota_\alpha Q,$$

and of course, the Poisson bracket can be recovered from this map. In general, the rank of this map is unconstrained. If it is constant, then Q is said to be of *constant rank*, and if the map is invertible, then Q is called *non-degenerate*. Note that the Hamiltonian vector field of f is given by $X_f = Q(df)$.

Example 2.2.2. The canonical example of a Poisson manifold is $\mathbb{R}^{2n} = \{(q_1, ..., q_n, p_1, ..., p_n)\}$, with bracket given by

$$\{f, g\} = \sum_i^n \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right),$$

for $f, g \in C^\infty(\mathbb{R}^{2n})$. The bivector for this Poisson structure is given by

$$Q = \sum_i^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

■

Example 2.2.3. The choice of a skew-symmetric $n \times n$ matrix A defines a *constant* Poisson structure on $\mathbb{R}^n$, defined by $\{x_i, x_j\} = A_{ij}$, for the standard coordinates $x_1, ..., x_n$. The Jacobi identity follows immediately from the fact that A is constant, because $\{x_k, \{x_i, x_j\}\} = \{x_k, A_{ij}\} = 0$. Furthermore, the map from the cotangent to the tangent bundle is essentially given by $-A$, and so the Poisson structure is of constant rank, with rank equal to the rank of A. ■

Recall that a *symplectic form* on a manifold M is given by a closed 2-form $\omega \in \Omega^2(M)$, such that the induced map

$$\omega : TM \to T^*M, \qquad X \mapsto \iota_X \omega,$$

is an isomorphism. Associated to a symplectic form ω, there is a naturally defined Poisson structure Q, defined by inverting the above bundle map: $Q = \omega^{-1}$. The Jacobi identity for Q follows from the closure of ω. Evidently, this Poisson structure is non-degenerate. In fact, all non-degenerate Poisson structures arise in this way from symplectic forms.

Proposition 2.2.4. *A Poisson structure Q is non-degenerate if and only if it is of the form $Q = \omega^{-1}$, for a symplectic form ω.*

The Darboux theorem for symplectic manifolds (M, ω) says that there is a choice of coordinates $(q_1, ..., q_n, p_1, ..., p_n)$ for M, such that the form can be written as

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Computing the associated Poisson bracket, we find that

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{i,j},$$

and hence we recover the expression for the Poisson structure of Example 2.2.2.

Example 2.2.5. The next obvious example of a Poisson manifold is the zero Poisson structure, defined by setting $Q = 0$ on any smooth manifold. ■

Example 2.2.6. There is a natural Poisson structure defined on the dual space $\mathfrak{g}^*$ of a Lie alebra $(\mathfrak{g}, [-,-])$. The elements of $\mathfrak{g}$ are linear functions on $\mathfrak{g}^*$, and we define their Poisson bracket to be given by the Lie bracket, namely

$$\{f, g\} = [f, g],$$

where $f, g \in \mathfrak{g}$. The Poisson bracket is then defined by extending this bracket to all functions on $\mathfrak{g}^*$, using the Leibniz rule. To be more explicit, choose a basis $e_1, ..., e_n$ of $\mathfrak{g}$, and let C_{ij}^k be the corresponding structure constants, defined by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k.$$

This basis defines coordinates $\{x_i\}$ on $\mathfrak{g}^*$, via $x_i(\alpha) = \alpha(e_i)$. Then, in terms of these coordinates, the Poisson bivector can be written as

$$Q = \frac{1}{2} \sum_{i,j,k} C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The Poisson structure of this example is *linear*, meaning that the linear functions are closed under the Poisson bracket. It turns out that linear Poisson brackets on a vector space V are equivalent to Lie algebra structures on the dual space V^*. Because of this fact, it is sometimes said that Poisson manifolds give a non-linear generalization of Lie algebras. ■

Holomorphic Poisson brackets

The definition of a Poisson bracket easily generalizes to the holomorphic setting. Namely, a holomorphic Poisson bracket on a complex manifold X is a Lie bracket on the sheaf of holomorphic functions

$$\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X,$$

which satisfies the Leibniz rule. It corresponds to a holomorphic bivector σ satisfying the identity $[\sigma, \sigma] = 0$. Note that this implies in particular that σ is of type $(2,0)$, with respect to the (p,q)-decomposition on bivectors, and can be written in terms of its real and imaginary parts as follows

$$\sigma = \sigma_R + i\sigma_I,$$

for σ_R and σ_I real Poisson structures, satisfying $\sigma_R = I \circ \sigma_I$, for I the complex structure on X.

The previous examples of Poisson structures all generalize to the holomorphic setting. In particular, a holomorphic linear Poisson structure is given by a complex Lie algebra, and a non-degenerate holomorphic Poisson structure corresponds to a holomorphic symplectic form. Note that a holomorphic symplectic form is given by a holomorphic, closed $(2,0)$-form, which decomposes as

$$\Omega = B + i\omega,$$

for real symplectic forms B and ω, satisfying $B = \omega \circ I$. Therefore, we see that $I = \omega^{-1} \circ B$: a holomorphic symplectic form determines its own underlying complex structure.

Let's consider more examples of Poisson structures.

Example 2.2.7. Consider $\mathbb{C}^2 = \{(x,y)\}$, and define the Poisson structure to be

$$\sigma = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}.$$

This is an example of a linear Poisson structure, corresponding to the Lie algebra $\mathfrak{g} = span_{\mathbb{C}}(x,y)$, with $[y,x] = x$. ■

Example 2.2.8. Another example of a Poisson structure on $\mathbb{C}^2$ is given by

$$\sigma = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

This Poisson structure has many symmetries: for example, the scaling action by $\mathbb{C}^* \times \mathbb{C}^*$ gives an action by Poisson isomorphisms. Let us consider the subgroup $\mathbb{Z} \subseteq \mathbb{C}^* \times \mathbb{C}^*$ generated by $(2,2)$. The Poisson structure descends to the quotient $X = (\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{Z}$, which is an example of a Hopf surface. The locus $y = 0$ in the quotient is given by $\mathbb{C}^*/\mathbb{Z}$, which is an elliptic curve, and the case for $x = 0$ is identical. Therefore, we get a Poisson structure on the Hopf surface, which drops rank on the union of two elliptic curves. ■

Example 2.2.9. Let X be a complex surface, and let $\sigma \in \Gamma(\wedge^2 TX)$ be a holomorphic bivector. This automatically defines a Poisson structure, because $[\sigma, \sigma]$ is a 3-vector, and so must vanish for dimensional reasons. In fact, $\wedge^2 TX = K_X^{-1}$, the anti-canonical line bundle, and the vanishing locus for σ is called the *anti-canonical* class of X. Let us consider the concrete example of $X = \mathbb{P}^2$. In this case, $K_X^{-1} \cong \mathcal{O}(3)$, and so σ vanishes on a cubic curve, which, for a generic choice of σ, is an elliptic curve. See [90] for more details on Poisson surfaces. ■

Note that examples 2.2.7 and 2.2.8 arise as restrictions of Poisson structures on $\mathbb{P}^2$ to affine open subsets. Namely, $x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}$ extends to a Poisson structure on $\mathbb{P}^2$ which vanishes to order 1 along a hyperplane, and to order 2 along another hyperplane. And $xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ extends to a Poisson structure which vanishes to order 1 along the three coordinate hyperplanes.

2.2.1 Poisson Lie algebroids and Symplectic groupoids

There is a natural Lie algebroid structure on the cotangent bundle of any Poisson manifold (M, Q), which we will denote $T^*_Q M$. The anchor map is given by $Q : T^*M \to TM$, and the bracket is given by

$$[\alpha, \beta] = \mathcal{L}_{Q(\alpha)}(\beta) - \mathcal{L}_{Q(\beta)}(\alpha) - dQ(\alpha, \beta),$$

for 1-forms α and β, where $\mathcal{L}$ denotes the Lie derivative. In the case of exact 1-forms, the bracket simplifies to

$$[df, dg] = d\{f, g\}.$$

Since exact forms generate the cotangent bundle, checking the axioms of a Lie algebroid reduces to checking them on exact forms, in which case they follow easily from the axioms of Poisson geometry.

The Poisson manifold giving rise to the Lie algebroid structure on T^*M is genuinely extra data, and not all Lie algebroid structures on the cotangent bundle arise in this way. It is therefore natural to ask whether there is a corresponding integrated structure on the Lie groupoids integrating $T^*_Q M$. This is indeed the case: Poisson manifolds are the infinitesimal counterpart to *symplectic groupoids*, a notion which is due to Karasëv, Weinstein and Zakrzewski [63, 109, 115, 116].

Definition 2.2.10 (Symplectic groupoid)**.** A symplectic groupoid $(\mathcal{G} \rightrightarrows M, \Omega)$ is a Lie groupoid $\mathcal{G}$ equipped with a *multiplicative symplectic form* Ω, which is a symplectic form satisfying the following equation on the fibre product $\mathcal{G} \times_M \mathcal{G}$

$$m^*\Omega = p_1^*\Omega + p_2^*\Omega,$$

where $m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ is the groupoid multiplication, and $p_i : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ are the two projections.

Many of the fundamental properties of symplectic groupoids were established in the papers cited above, as well as [24, 78]. We review some of these in the following.

There is a useful reformulation of the definition of a multiplicative symplectic form, which makes use of the notion of *Lagrangian submanifolds*. Recall that a Lagrangian submanifold of a $2n$-dimensional symplectic manifold (M, ω), is an n-dimensional submanifold $\iota : L \to M$, such that $\iota^*\omega = 0$. A symplectic form Ω on a Lie groupoid $\mathcal{G}$ is multiplicative if and only if the graph of the multiplication map is a Lagrangian submanifold of $(\mathcal{G}, \Omega) \times (\mathcal{G}, -\Omega) \times (\mathcal{G}, -\Omega)$.

Let us note a few important properties which are satisfied by symplectic groupoids. First, $(\mathcal{G}, \Omega)$ determines a unique Poisson structure σ on the manifold of objects B, such that the target t is Poisson, and the source s is anti-Poisson, where the Poisson structure on $\mathcal{G}$ is the one determined by Ω. The Lie algebroid of $\mathcal{G}$ is then isomorphic to the cotangent Lie algebroid $T^*_\sigma B$. Furthermore, the identity bisection is Lagrangian, the inversion map is anti-Poisson, and the source and target fibres are *symplectic orthogonal*, which means that they are orthogonal complements, with respect to the pairing on tangent vectors given by Ω.

A Poisson structure σ is said to be *integrable* if its corresponding Lie algebroid is integrable. In this case, the source-simply connected integrating groupoid inherits a unique multiplicative symplectic form, such that the Poisson structure it determines on the manifold of objects coincides with σ [75, 27]. This groupoid is called the *Weinstein groupoid*, and is denoted $\Sigma(M)$. In this thesis, all Poisson structures we consider are assumed to be integrable. Note that by the work of Laurent-Gengoux, Stiénon and Xu [68], a holomorphic Poisson manifold is integrable if and only if either its real, or imaginary part is integrable.

The tangent spaces to the orbits of a symplectic groupoid are generated by the Hamiltonian vector fields, and the orbits inherit a symplectic structure. Hence, they are called the *symplectic leaves* of the Poisson manifold. In fact, the symplectic leaves exist, regardless of whether the Poisson manifold is integrable. They are given by the equivalence classes of points related by Hamiltonian flows.

Bisections

For a symplectic groupoid, the natural bisections to consider are Lagrangian submanifolds, defining the *Lagrangian bisections*. These form a subgroup of the group of bisections, $LBis(\mathcal{G}, \Omega)$, and they act on $(\mathcal{G}, \Omega)$ via symplectomorphisms (i.e. diffeomorphisms preserving Ω). More precisely, we have the following proposition.

Proposition 2.2.11. *Let $\lambda : B \to (\mathcal{G}, \Omega)$ be a Lagrangian bisection, viewed as a section of the source map. Then $\phi = t \circ \lambda$ is a Poisson isomorphism of the base Poisson manifold, and the left, right and conjugation maps preserve the symplectic form Ω.*

Proof. First, note that the three properties listed above, namely that t is Poisson, s is anti-Poisson, and the source and target fibres are symplectic orthogonal, imply that

$$(t \times s) : (\mathcal{G}, \Omega) \to (M, \sigma) \times (M, -\sigma),$$

is Poisson. Let $L = \mathrm{im}(\lambda)$ be the bisection viewed as a Lagrangian submanifold. The image $(t \times s)(L) \subseteq M \times M$ is the graph of the diffeomorphism ϕ, and, as one can check, it is Poisson as a result of L being Lagrangian.

Consider now the left multiplication map $L_\lambda : \mathcal{G} \to \mathcal{G}$. The map

$$\mathcal{G} \to (\mathcal{G}, \Omega) \times (\mathcal{G}, -\Omega) \times (\mathcal{G}, -\Omega), \qquad g \mapsto (L_\lambda(g), \lambda(t(g)), g),$$

factors through the graph of the multiplication, and hence the symplectic form on the triple product pulls back to 0. Therefore

$$0 = L_\lambda^*\Omega - t^*\lambda^*\Omega - \Omega = L_\lambda^*\Omega - \Omega,$$

since λ is Lagrangian. This verifies that L_λ preserves Ω. The case of right multiplication and conjugation follow analogously. $\square$

Recall that the Lie algebra of the group of bisections is given by global sections of the Lie algebroid $\Gamma(T^*_Q M) = \Omega^1(M)$. In this case, the construction of the exponential map is simplified by the presence of the symplectic form Ω.

Lemma 2.2.12. *Given a 1-form $\alpha \in \Omega^1(M)$, the associated right-invariant vector field on $\mathcal{G}$ is given by $\Omega^{-1}(t^*\alpha)$.*

Proof. We first of all show that $X_\alpha = \Omega^{-1}(t^*\alpha)$ is tangent to the source fibres. Since the source and target fibres are symplectic orthogonal, it follows that $ds \circ \Omega^{-1} \circ dt^* = 0$, and hence $ds(X_\alpha) = 0$. Next, we show that X_α is right invariant, meaning that for composable pairs (h, g), we have $dR_g(X_\alpha(h)) = X_\alpha(hg)$. Note that this is well-defined because X_α is tangent to the source fibres. Let μ be a local Lagrangian bisection passing through g, viewed as a section of the target map. Then by proposition 2.2.11, right multiplication by μ preserves Ω, and hence also its inverse. Therefore

$$dR_g(X_\alpha(h)) = dR_\mu|_h(\Omega^{-1}(t^*\alpha)|_h) = \Omega^{-1}((t \circ R_{\mu^{-1}})^*\alpha)|_{hg} = X_\alpha(hg),$$

where we use the fact that $t \circ R_{\mu^{-1}} = t$. Finally, we show that X_α agrees with α along the identity bisection. The identification between $\ker(ds)$ and T^*M is provided by Ω. Since the identity bisection is Lagrangian, Ω gives a perfect pairing between $\ker(ds)_x$ and TM_x, and hence defines an isomorphism

$$\Omega|_{\ker(ds)_x} : \ker(ds)_x \to T^*M_x.$$

Under this identification, we see that X_α and α agree. □

Given a 1-form $\alpha \in \Omega^1(M)$, we obtain its 1-parameter subgroup by taking the flow ϕ_t of the vector field $X_\alpha = \Omega^{-1}(t^*\alpha)$, and applying it to the identity bisection ϵ. This gives the family of bisections $\exp(t\alpha) = \phi_t \circ \epsilon$. By lemma 2.1.23, the flow ϕ_t is then given by left multiplication by $\exp(t\alpha)$. If α is closed, then ϕ_t acts by symplectomorphisms, and so $\exp(t\alpha)$ is a family of Lagrangian bisections. Conversely, if the bisections are Lagrangian, then by proposition 2.2.11 the flow acts by symplectomorphisms, and hence α must be closed. We therefore obtain the Lie algebra of the group of Lagrangian bisections.

Proposition 2.2.13. *[109, 113] The Lie algebra of the group of Lagrangian bisections is given by the closed 1-forms on M, with bracket determined by Q.*

$$Lie(LBis(\mathcal{G}, \Omega)) = \Omega^{1,cl}(M).$$

Examples

Example 2.2.14. Consider the zero Poisson structure $\sigma = 0$ on a manifold M. The corresponding Poisson Lie algebroid is given by T^*M, with zero anchor and bracket. This is a special case of Example 2.1.20, and hence, the symplectic groupoid is given by T^*M, with its canonical symplectic form Ω_0.

Let us recall how this form is defined. Given coordinates (x_i) on M, we get natural coordinates (x_i, p_i) on T^*M, such that (x_i, p_i) corresponds to the 1-form $\sum_i p_i dx_i$. The canonical symplectic form is defined in these coordinates to be

$$\Omega_0 = \sum_i dp_i \wedge dx_i.$$

We can see that it is multiplicative in the following way. The space of composable arrows is given by $T^*M \times_M T^*M$, and so has coordinates $(x_i, p_i^{(1)}, p_i^{(2)})$. In these coordinates, the multiplication map is given by

$$m(x_i, p_i^{(1)}, p_i^{(2)}) = (x_i, p_i^{(1)} + p_i^{(2)}),$$

and the two projection maps are given by

$$p_k(x_i, p_i^{(1)}, p_i^{(2)}) = (x_i, p_i^{(k)}).$$

Therefore,

$$m^*(\sum_i dp_i \wedge dx_i) = \sum_i d(p_i^{(1)} + p_i^{(2)}) \wedge dx_i = p_1^*(\sum_i dp_i \wedge dx_i) + p_2^*(\sum_i dp_i \wedge dx_i).$$

■

Example 2.2.15. Let (M, ω) be a symplectic manifold, and consider the corresponding Poisson structure. The anchor map of the Poisson Lie algebroid is given by ω^{-1}, which is invertible. Therefore, the Poisson Lie algebroid is isomorphic to the tangent algebroid TM, and so its source simply connected integration is given by the fundamental groupoid $\Pi(M)$.

Lets instead consider the pair groupoid $Pair(M)$, which is another integration, and try to determine its multiplicative symplectic form. This is a symplectic form Ω on the product $M \times M$, satisfying the property that the two projection maps to M are respectively Poisson, and anti-Poisson. One possibility for the symplectic form on the groupoid is therefore given by $(\omega, -\omega) = t^*\omega - s^*\omega$. Let us check that this form is indeed multiplicative. First note that the diagonal embedding $\Delta : M \to (M, \omega) \times (M, -\omega)$ is Lagrangian, since the symplectic form pulls back to $\omega - \omega = 0$. Next, the space of composable arrows is given by $M \times M \times M$, with multiplication and projection maps given by the 3 projections to $M \times M$. It is then straightforward to compute that the graph of the multiplication map is given by 3 copies of the diagonal embedding:

$$\Delta^3 : M^3 \to ((M, \omega) \times (M, -\omega))^3,$$

and hence is Lagrangian.

Since the groupoid morphism $(t \times s) : \Pi(M) \to Pair(M)$ is a covering space projection, we get the multiplicative symplectic form on $\Pi(M)$ simply by pulling back the form from $Pair(M)$. It is therefore given by $\Omega = t^*\omega - s^*\omega$. ■

Example 2.2.16. Let $\mathfrak{g}$ be a Lie algebra, and consider the Poisson structure on the dual space $\mathfrak{g}^*$. The cotangent bundle is given by $T^*(\mathfrak{g}^*) = \mathfrak{g} \times \mathfrak{g}^*$, and one can check that the Lie algebroid structure is that of an action algebroid, for the coadjoint action $ad^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$. Following Example 2.1.21, we therefore see that the integrations are given by action groupoids, associated to the coadjoint actions of any integrating Lie groups G. In order to see the symplectic structure, note that the space of arrows of the action groupoid is given by $G \times \mathfrak{g}^*$, which is isomorphic to the cotangent bundle T^*G. One then checks that the source and target maps are given by left and right trivialization respectively, and the multiplication is given by the following formula:

$$\xi_{g_1} \star \eta_{g_2} = d(R_{g_2^{-1}})^*_{g_1 g_2}(\xi_{g_1}) = d(L_{g_1^{-1}})^*_{g_1 g_2}(\eta_{g_2}),$$

where R_g and L_g are right and left multiplication, respectively. The multiplicative symplectic form is then given simply by the canonical symplectic form on T^*G. See [24] for further details. ∎

Example 2.2.17. Consider the Poisson structure given by $\sigma = x\partial_y \wedge \partial_x$ on $\mathbb{C}^2$. As we noted before, this is a linear Poisson structure associated to the Lie algebra $\mathfrak{g} = span_{\mathbb{C}}(x, y)$, with $[y, x] = x$. As such, we can obtain the integration of this Poisson bracket as a special instance of the previous example. Indeed, the Lie group integrating this algebra is the group G of affine transformations of $\mathbb{C}$ (or rather the universal cover of this, if we want to produce the Weinstein groupoid). Therefore we can obtain the symplectic groupoid integrating our Poisson structure as the action groupoid for the coadjoint action of G on $\mathfrak{g}^*$.

However, in the present case a more direct approach to integration is possible: note that the Hamiltonian vector fields corresponding to the coordinate functions are given by $X_{-x} = x\partial_y$ and $X_y = x\partial_x$. The vector field $x\partial_x$ generates the multiplicative action of $\mathbb{C}$: $(a, x) : x \mapsto e^a x$, and the vector field $x\partial_y$ generates the additive action of $\mathbb{C}$ rescaled by a factor of x: $(b, y) : y \mapsto y + xb$. Combining the two actions we get the action groupoid $\mathcal{G} = \mathbb{C}^2 \ltimes \mathbb{C}^2$, with coordinates (a, b, x, y), with the source and target maps given respectively by

$$s(a, b, x, y) = (x, y) \qquad t(a, b, x, y) = (e^a x, y + xb),$$

and with the multiplication given by

$$(a_1, b_1, x_1, y_1) \star (a_2, b_2, x_2, y_2) = (a_1 + a_2, b_1 e^{a_2} + b_2, x_2, y_2).$$

Note that indeed $\mathcal{G} = \widetilde{\mathrm{Aff}(\mathbb{C})} \ltimes \mathbb{C}^2$: the action groupoid for the action of (the universal cover of) the affine group on the dual of its Lie algebra.

To compute the symplectic form on $\mathcal{G}$ we make use of the following fact: the Poisson structure σ is non-degenerate on an open dense subset of $\mathbb{C}^2$, given by the locus of points where $x \neq 0$. We can therefore invert it to get a meromorphic symplectic form

$$\omega = \frac{1}{x} dx \wedge dy.$$

Over the locus $x \neq 0$, we know from Example 2.2.15 that the multiplicative symplectic form on the groupoid is given by $\Omega = t^*\omega - s^*\omega$. And because this equation holds over a dense subset of $\mathcal{G}$, it must hold over the entire groupoid. Computing this form, we find

$$\begin{aligned}
\Omega &= t^*\omega - s^*\omega \\
&= \frac{1}{e^a x} d(e^a x) \wedge d(y + xb) - \frac{1}{x} dx \wedge dy \\
&= da \wedge d(y + xb) - db \wedge dx.
\end{aligned}$$

See [92] for further discussion of this example in the smooth category. ∎

Example 2.2.18. Consider the Poisson structure given by $\sigma = xy\partial_y \wedge \partial_x$ on $\mathbb{C}^2$. This Poisson structure is not linear, and its Lie algebroid does not arise from an action. However, it is still possible to integrate this Poisson structure, using the *spray construction* [29, 17]. We will not explain this construction in general, but will instead focus on this specific example. Note that this example is a special case of a construction in [69].

Consider the cotangent bundle $T^*\mathbb{C}^2$, which has coordinates (a, b, x, y) corresponding to the 1-form

$adx + bdy$, and consider the following Poisson spray vector field

$$X = ybx\partial_x - xay\partial_y - yba\partial_a + xab\partial_b.$$

The salient features of this vector field are that it is of weight 1 with respect to the scaling action on the cotangent fibres, and it satisfies the following equation:

$$\sigma^\sharp(\alpha) = (d\pi)_\alpha(X_\alpha),$$

for $\alpha \in T^*\mathbb{C}^2$, where $\pi : T^*\mathbb{C}^2 \to \mathbb{C}^2$ is the bundle projection. It it straightforward to integrate this vector field: the flow is given by

$$(a(t), b(t), x(t), y(t)) = (a_0 \exp(-ybt), b_0 \exp(xat), x_0 \exp(ybt), y_0 \exp(-xat)).$$

The flow construction gives a way of constructing a symplectic groupoid structure on $T^*\mathbb{C}^2$ using the flow of this vector field. In particular, the source map is given by the bundle projection $s(a, b, x, y) = (x, y)$, and the target map is given by the time 1-flow, followed by the vector bundle projection, so that $t(a, b, x, y) = (x \exp(yb), y \exp(-xa))$. From this, it is possible to work out the multiplication, and we see that it is given by

$$(a_1, b_1, x_1, y_1) \star (a_2, b_2, x_2, y_2) = (e^{y_2 b_2} a_1 + a_2, e^{-x_2 a_2} b_1 + b_2, x_2, y_2).$$

The multiplicative symplectic form also follows from the flow construction, but in this case it is easier to proceed as in the previous example, since σ is non-degenerate on an open dense subset of $\mathbb{C}^2$. Inverting σ, we get the following meromorphic form

$$\omega = \frac{dx \wedge dy}{xy},$$

and hence the form on the groupoid is given by

$$\Omega = t^*\omega - s^*\omega = da \wedge dx + db \wedge dy + d(xa) \wedge d(yb).$$

■

Example 2.2.19. Consider the Poisson structure σ on $\mathbb{CP}^2$ which vanishes to order 1 on the three coordinate hyperplanes. Lets us choose projective coordinates $[x : y : z]$, such that these hyperplanes are given by $[0 : y : z]$, $[x : 0 : z]$ and $[x : y : 0]$. The 0-dimensional symplectic leaves of this Poisson structure are given by the points of these hyperplanes, and the unique 2-dimensional symplectic leaf is given by the locus $xyz \neq 0$, which is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. The restriction of this Poisson structure to the complement of each hyperplane coincides with the example above. Therefore, the symplectic groupoid of σ can be constructed by gluing three copies of the groupoid of the previous example, along the symplectic leaf $\mathbb{C}^* \times \mathbb{C}^*$. In Section 5.2, we will give a different description of this symplectic groupoid.

A generic Poisson structure on $\mathbb{CP}^2$ is an example of a *log symplectic* Poisson structure. In general, a log symplectic Poisson structure is given by a Poisson structure σ on a $2n$-dimension manifold, such that the top exterior power σ^n vanishes transversely. A systematic method for constructing integrations of these Poisson manifolds has been developed in [50]. ■

2.2.2 Symplectic Morita Equivalence

The definition of Morita equivalence can be upgraded to include the data of a symplectic form, thereby giving rise to the notion of Morita equivalence between symplectic groupoids [112] and Poisson manifolds [111], both due to Xu. These definitions generalize Morita equivalence between Lie groupoids considered in Section 2.1.2, as well as the notions of symplectic realization, and symplectic dual pair, which are due to Weinstein [108].

Definition 2.2.20 (Symplectic Morita equivalence)**.** A *symplectic Morita equivalence* between symplectic groupoids $(\mathcal{G}, \omega_1) \rightrightarrows B$ and $(\mathcal{H}, \omega_2) \rightrightarrows C$ is given by a symplectic manifold (S, ω), such that S is a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$, and ω is multiplicative with respect to the left and right actions. More precisely, the graph of the left $\mathcal{G}$ action is a Lagrangian submanifold of $(\mathcal{G}, -\omega_1) \times (S, -\omega) \times (S, \omega)$, and the graph of the right $\mathcal{H}$ action is a Lagrangian submanifold of $(S, \omega) \times (S, -\omega) \times (\mathcal{H}, -\omega_2)$.

Definition 2.2.21 (Morita equivalence of Poisson manifolds)**.** A Morita equivalence between Poisson manifolds (M_1, Q_1) and (M_2, Q_2) is a symplectic manifold (S, ω), together with a pair of surjective submersions $\pi_1 : S \to M_1, \pi_2 : S \to M_2$, with connected and simply-connected fibres, such that

1. π_1 is Poisson and π_2 is anti-Poisson;
2. the vertical distributions $\ker(d\pi_1)$ and $\ker(d\pi_2)$ are symplectically orthogonal;
3. π_1, π_2 are *complete* in the sense that the pullback of any complete Hamiltonian vector field is complete.

We view this as a morphism from (M_2, Q_2) to (M_1, Q_1).

Remark 2.2.22. This definition makes use of the notion of the pullback of a Hamiltonian vector field. This is defined as follows. Let $\pi : (S, Q) \to (M, Q')$ be a Poisson map. The pullback of a Hamiltonian vector field $X_f = Q'(df)$ is defined to be $\pi^* X_f := X_{\pi^* f} = Q(\pi^* df)$. Note that since π is Poisson, it follows that $\pi^* X_f$ is π-related to X_f.

The notion of Morita equivalence between Poisson manifolds has the advantage that it makes no reference to symplectic groupoids. However, Xu has shown that in fact, the two definitions are very closely related.

Theorem 2.2.23. *[111, Theorem 3.2] Two integrable Poisson manifolds (M_1, Q_1) and (M_2, Q_2) are Morita equivalent if and only if their Weinstein groupoids $\Sigma(M_1)$ and $\Sigma(M_2)$ are symplectically Morita equivalent.*

Similarly to the case of groupoids, we can define a bicategory **sStack**, whose objects are integrable Poisson manifolds (or equivalently, source simply connected symplectic groupoids), 1-morphisms are symplectic Morita equivalences, and 2-morphisms are isomorphisms of bibundles intertwining all the structure. The composition of symplectic Morita equivalences is defined in the same way as for ordinary Morita equivalences (see Section 2.1.2), and the symplectic form is obtained via symplectic reduction [112]. An object (M, σ) of this category represents the quotient stack $[M/\Sigma(M)]$, which is the space of orbits, or symplectic leaves, of the Poisson structure. The Weinstein groupoid $\Sigma(M)$ gives the identity morphism of a Poisson manifold. See [15, 16, 66] for more details of this category.

Viewing a symplectic groupoid $(\mathcal{G}, \omega)$ as an object of **dStack**, it is natural to ask how the symplectic form ω should be interpreted. The answer, according to [104, 93], is that the symplectic form ω gives rise to a degree +1 symplectic form on $[B/\mathcal{G}]$. In the language of [85], the quotient stack $[B/\mathcal{G}]$ is 1-*shifted symplectic*, and the natural map $B \to [B/\mathcal{G}]$ carries a *Lagrangian structure*, which gives rise to the Poisson structure σ [91]. Furthermore, the intersection of two Lagrangians in a 1-shifted symplectic structure gives rise to a usual (0-shifted) symplectic structure, which in the case of $\mathcal{G} = B \times_{[B/\mathcal{G}]} B$ is the symplectic form ω on the total space of the groupoid. See also [103] for another account of the stack associated to a symplectic groupoid.

Examples

Example 2.2.24. The trivial Morita self-equivalence of a Poisson manifold (M, σ) is the Weinstein groupoid $\Sigma(M)$. It has a Lagrangian bisection given by the identity bisection. ■

Example 2.2.25. Let (S_1, ω_1) and (S_2, ω_2) be simply connected symplectic manifolds. Then

$$\begin{tikzcd} & (S_1,\omega_1)\times(S_2,-\omega_2) \arrow[dl] \arrow[dr] & \\ S_1 & & S_2 \end{tikzcd}$$

defines a Morita equivalence.

More generally, consider a symplectic manifold (S, ω). It's Weinstein groupoid is the fundamental groupoid $\Pi(S)$, with symplectic form $t^*\omega - s^*\omega$. The Morita equivalence between $\Pi(S)$ and the fundamental group, $\pi_1(S, s)$, can be upgraded to include the symplectic data. It is given by the universal cover $\pi : \tilde{S} \to S$, with symplectic form $\pi^*\omega$. Because Morita equivalence is an equivalence relation, it then follows that two symplectic manifolds are Morita equivalent if and only if they have the same fundamental group. See [111] for details. ■

Example 2.2.26. Morita equivalences can be composed with Poisson diffeomorphisms [15]. Let $\phi : M_2 \to M_1$ be a Poisson diffeomorphism and let $M_2 \xleftarrow{\pi_2} S \xrightarrow{\pi_3} M_3$ be a Morita equivalence. Then we get the following Morita equivalence

$$\begin{tikzcd} & S \arrow[dl, "\phi\circ\pi_2"'] \arrow[dr, "\pi_3"] & \\ M_1 & & M_3 \end{tikzcd}$$

In particular, by composing Poisson diffeomorphisms with the trivial Morita equivalence, we get a map from Poisson diffeomorphisms to Morita equivalences. This corresponds to a functor from the category of Poisson manifolds, and Poisson diffeomorphisms, to the bicategory **sStack**. ■

Example 2.2.27. Let (M, Q) be a real Poisson manifold. If $B \in \Omega^{2,cl}(M)$ is a closed 2-form such that $id + B \circ Q : T^*M \to T^*M$ is invertible, then we can define a new Poisson structure

$$Q^B := Q \circ (id + B \circ Q)^{-1},$$

which is called the *B-field transform* of Q [95]. Bursztyn and Radko show in [14] that in this case (M, Q)

and (M, Q^B) are Morita equivalent in the following way

$$\begin{tikzcd}
& (\Sigma(M), \Omega + t^*B) \arrow[dl, "t"'] \arrow[dr, "s"] & \\
(M, Q^B) & & (M, Q)
\end{tikzcd}$$

where $(\Sigma(M), \Omega)$ is the Weinstein groupoid of (M, Q) and s and t are the source and target maps, respectively. Note that the identity bisection of the Weinstein groupoid gives rise to a non-Lagrangian bisection of the equivalence. Bursztyn and Radko also show that the Weinstein groupoid of (M, Q^B) is given by $\Sigma(M)$, but with the symplectic form modified to be $\Omega + t^*B - s^*B$.

If we apply this construction to the zero Poisson structure $(M, Q = 0)$ then we get a Morita self-equivalence given by $(T^*M, \Omega_0 + p^*B)$, where Ω_0 is the canonical symplectic form and $p : T^*M \to M$ is the projection. ■

Bisections

It makes sense to consider both ordinary bisections, as well as Lagrangian bisections of symplectic Morita equivalences. We can compose Morita equivalences equipped with bisections, and the equivalences with Lagrangian bisections are closed under this operation. Generalizations of the following proposition will play an important role in this thesis. Various versions appear in [24, 92, 13].

Proposition 2.2.28. *Let (S, ω) be a Morita equivalence between source simply connected symplectic groupoids $(\mathcal{G}, \omega_1) \rightrightarrows (B, P)$ and $(\mathcal{H}, \omega_2) \rightrightarrows (C, Q)$, and let $b : C \to S$ be a bisection, viewed as a section of $q : S \to C$. Let $\phi = p \circ b : C \to B$, where $p : S \to B$, and $F = b^*\omega$. Then*

1. *ϕ is a Poisson isomorphism between Q^F, the B-field transform of Q, and P;*
2. *the induced isomorphism of groupoids $\Phi : \mathcal{H} \to \mathcal{G}$ is the morphism which integrates $d\phi$, via Proposition 2.1.22, and it satisfies*
$$\Phi^*\omega_1 = \omega_2 + t^*F - s^*F;$$
3. *the induced isomorphism $T : \mathcal{H} \to S$ satisfies $T^*\omega = \omega_2 + t^*F$.*

In particular, if b is Lagrangian, then ϕ is a Poisson isomorphism between (C, Q) and (B, P), and the symplectic groupoids $(\mathcal{G}, \omega_1)$ and $(\mathcal{H}, \omega_2)$ are isomorphic.

Proof. The bisection b induces two diffeomorphisms

$$\begin{aligned} T :& \mathcal{H} \to \mathcal{S}, \qquad h \mapsto b(t(h))h, \\ R :& \mathcal{G} \to \mathcal{S}, \qquad g \mapsto gb(\phi^{-1} \circ s(g)). \end{aligned}$$

Then since
$$\mathcal{H} \to S \times S \times \mathcal{H}, \qquad h \mapsto (T(h), bt(h), h)$$

factors through the graph of the action, which is Lagrangian, we see that $T^*\omega = \omega_2 + t^*F$. Similarly $R^*\omega = \omega_1 + s^*(\phi^{-1})^*F$. Then since the induced groupoid isomorphism $\Phi : \mathcal{H} \to \mathcal{G}$ satisfies $\Phi = R^{-1} \circ T$, we have that $\Phi^*(\omega_1 + s^*(\phi^{-1})^*F) = \omega_2 + t^*F$, which we can rearrange to give

$$\Phi^*\omega_1 = \omega_2 + t^*F - s^*F.$$

Note that since ω is invertible, it follows that $\omega_2 + t^*F$ must also be invertible. Following [14], this implies that $id + F \circ Q$ is invertible, and that $t_*(\omega_2 + t^*F)^{-1} = Q^F$, the B-field transformation of Q by F. The relation $p \circ T = \phi \circ t$ then implies that $\phi_*(Q^F) = P$.

Recall from [14] that the symplectic groupoid integrating Q^F is given by $\mathcal{H}$, with symplectic form modified to $\omega_2 + t^*F - s^*F$. Hence, Φ is a morphism of symplectic groupoids between $(\mathcal{H}, \omega_F = \omega_2 + t^*F - s^*F)$ and $(\mathcal{G}, \omega_1)$, and it covers the Poisson isomorphism $\phi : (C, Q^F) \to (B, P)$. The map ϕ induces an isomorphism of the corresponding cotangent Lie algebroids, and we need to check that this morphism agrees with the one induced by Φ. To see this, consider a 1-form $\alpha \in \Omega^1(B)$, and its pullback $\phi^* \in \Omega^1(C)$. The right invariant vector field on $\mathcal{G}$ induced by α is given by $Y = \omega_1^{-1}(t^*\alpha)$, and the right invariant vector field on $\mathcal{H}$ induced by $\phi^*\alpha$ is given by $X = \omega_F^{-1}(t^*\phi^*\alpha)$. Then

$$\Phi_*(X) = \Phi_*\omega_F^{-1}t^*\phi^*(\alpha) = \Phi_*\omega_F^{-1}\Phi^*t^*\alpha = \omega_1^{-1}(t^*\alpha) = Y.$$

Therefore $Lie(\Phi) = d\phi$. $\square$

Chapter 3

Generalized Kähler geometry

In this section, we review Generalized Kähler (GK) geometry, and more specifically *GK structures of symplectic type*, which are the main object of study in this thesis. There is a dizzying variety of formulations of this geometry (see for example, [44, 48, 46, 49]), and the goal of this chapter is to quickly get from the original bihermitian formulation of [37], to the formulation introduced in [46], in terms of the data (I_+, I_-, Q, F), consisting of a pair of complex structures $I_\pm$, a Poisson structure Q, and a symplectic form F. In order to keep the prerequisites minimal, we omit from the presentation the calculus of Dirac geometry, and we do not discuss the B-field symmetries. For an equivalent exposition that includes these aspects, see [6, Section 1].

Definition 3.0.1 (Generalized Kähler Geometry)**.** A *Generalized Kähler structure* consists of a Riemannian manifold (M, g), and a pair of complex structures I_+ and I_-, such that

1. the metric g is Hermitian with respect to I_+ and I_-, meaning that

$$g(I_\pm v, I_\pm u) = g(v, u),$$

   for tangent vectors $v, u \in TM$;

2. the associated Hermitian forms $\omega_\pm = gI_\pm$ (which are of type $(1,1)$ with respect to their respective complex structure) satisfy the following integrability condition

$$d^c_+\omega_+ + d^c_-\omega_- = 0, \qquad dd^c_\pm\omega_\pm = 0, \tag{3.1}$$

   where $d^c_\pm = i(\bar{\partial}_\pm - \partial_\pm)$ are the real operators determined by the de Rham operator d and the complex structures.

The closed 3-form $H = d^c_+\omega_+$ is known in the physics literature as the Wess–Zumino–Witten (WZW) term of the non-linear sigma model.

Remark 3.0.2. A few technical comments about the operators $I_\pm$ and $d^c_\pm$. The complex structures $I_\pm$ interact with differential forms in several ways. So far, we have seen that it is possible to compose the vector bundle morphism associated to a metric, or a 2-form, with a complex structure, as in $\omega_+ = gI_+$. However, a complex structure may also act on the graded algebra of differential forms as both a derivation and an automorphism. Let us denote by $I_{\pm}^{a}$, the automorphism action of the complex structure on differential forms, which satisfies

$$I_{\pm}^{a}(\alpha \wedge \beta) = (I_{\pm}^{a}\alpha) \wedge (I_{\pm}^{a}\beta),$$

and by $I_{\pm}^{d}$, the derivation action, which satisfies

$$I_{\pm}^{d}(\alpha \wedge \beta) = (I_{\pm}^{d}\alpha) \wedge \beta + \alpha \wedge (I_{\pm}^{d}\alpha).$$

It is then straightforward to check that in terms of these two actions, the $d_{\pm}^{c}$ operators have the following expressions

$$d_{\pm}^{c} = (I_{\pm}^{a})^{-1} d I_{\pm}^{a} = [d, I_{\pm}^{d}].$$

As a special case of the definition of a GK structure, we can set $I_{+} = I_{-} = I$. The integrability condition 3.1 then reduces simply to $d\omega = 0$, and hence (g, I, ω) defines an ordinary Kähler structure. As is well-known, an equivalent description of this integrability condition, expressed in terms of the complex structure, says that

$$\nabla I = 0,$$

where ∇ is the Levi-Civita connection, which is the unique torsion free connection preserving the metric. We would like a similar reformulation of equation 3.1 for the general case. However, the complex structures $I_{\pm}$ in a generalized Kähler structure are usually not preserved by the Levi-Civita connection. Instead, we consider the connections

$$\nabla^{\pm} = \nabla \mp \frac{1}{2} g^{-1} H,$$

which have skew torsion given by $\mp g^{-1}H$.

Lemma 3.0.3. *The connections $\nabla^{\pm}$ preserve the metric g.*

Proof. The lemma is equivalent to the statement that for vector fields $X, Y, Z \in \mathfrak{X}(M)$, the following equation is satisfied

$$Xg(Y, Z) = g(\nabla_{X}^{\pm}(Y), Z) + g(Y, \nabla_{X}^{\pm} Z).$$

Expanding the right hand side, and using the fact that ∇ preserves the metric, and H is skew-symmetric, we get

$$g(\nabla_{X}^{\pm}(Y), Z) + g(Y, \nabla_{X}^{\pm} Z) = g(\nabla_{X}(Y), Z) + g(Y, \nabla_{X} Z) \mp \frac{1}{2}(H(X, Y, Z) + H(X, Z, Y)) = Xg(Y, Z).$$

□

Before proving the next result, we state the following identity, which holds for any Hermitian manifold (g, I), with corresponding $(1,1)$ form $\omega = gI$:

$$2g((\nabla_{X} I)Y, Z) = d\omega(X, Y, Z) - d\omega(X, IY, IZ), \tag{3.2}$$

where ∇ is the Levi-Civita connection. Note also that a 3-form H, of type $(2,1) + (1,2)$, as in the definition of a GK structure, satisfies the following identity

$$H(X, Y, Z) = H(IX, IY, Z) + H(IX, Y, IZ) + H(X, IY, IZ),$$

for vectors X, Y, Z.

Lemma 3.0.4. *The connections $\nabla^{\pm}$ preserve the respective complex structures $I_{\pm}$.*

Proof. We only show that ∇^{+} preserves I_{+}, as the other case works identically. The pair (g, I_{+}) is Hermitian, with Hermitian form ω_{+}, and so equation 3.2 implies the following identity for $\nabla^{+}(I_{+})$

$$\begin{aligned}2g((\nabla^{+}_{X} I_{+})Y, Z) &= 2g((\nabla_{X} I_{+})Y, Z) - (H(X, I_{+}Y, Z) + H(X, Y, I_{+}Z)) \\ &= d\omega_{+}(X, Y, Z) - d\omega_{+}(X, I_{+}Y, I_{+}Z) - H(X, I_{+}Y, Z) - H(X, Y, I_{+}Z),\end{aligned}$$

for vectors X, Y, Z. Next, since ω_{+} is of type $(1,1)$, we have that $I^{a}_{+}\omega_{+} = \omega_{+}$, and therefore $d\omega_{+} = I^{a}_{+} d^{c}_{+} \omega_{+} = I^{a}_{+} H$. Substituting this into the above equation, and using the fact that H is of type $(2,1) + (1,2)$ with respect to I_{+}, we find

$$2g((\nabla^{+}_{X} I_{+})Y, Z) = H(I_{+}X, I_{+}Y, I_{+}Z) - H(I_{+}X, Y, Z) - H(X, I_{+}Y, Z) - H(X, Y, I_{+}Z) = 0.$$

□

We conclude that $\nabla^{\pm}$ are the *Bismut connections* for $(g, I_{\pm})$, which are the unique connections with skew torsion, and which preserve both the metric and the complex structure (see [7] for more details). Equation 3.1 can be reformulated in terms of the Bismut connections of (g, I_{+}, I_{-}) as follows.

Proposition 3.0.5. *[44, Section 6] Given a metric g, which is Hermitian with respect to two complex structures $I_{\pm}$, equation 3.1 is equivalent to the condition that the Bismut connections of $(g, I_{\pm})$ have the form $\nabla^{\pm} = \nabla \mp \frac{1}{2} g^{-1} H$, for a closed 3-form H, where ∇ is the Levi-Civita connection.*

3.1 Symplectic Type.

Let us now restrict to the special case of GK structures of symplectic type.

Definition 3.1.1 (Generalized Kähler Geometry of Symplectic Type)**.** A GK structure (g, I_{+}, I_{-}) is of *symplectic type* if $I_{+} + I_{-}$ is invertible.

Remark 3.1.2. We will denote the inverse of $I_{+} + I_{-}$ by U.

The terminology of symplectic type comes from the fact that, as we will presently show, we can define a symplectic form, via the equation

$$g = -\frac{1}{2} F(I_{+} + I_{-}). \tag{3.3}$$

Lets start by establishing the main properties of F.

Lemma 3.1.3. *The tensor F is a closed 2-form, and satisfies the identity*

$$F I_{+} + I^{*}_{-} F = 0. \tag{3.4}$$

Since F is invertible by definition, it therefore defines a symplectic form on M.

Proof. Proving skew-symmetry of F amounts to showing that

$$g(X, (I_+ + I_-)X) = 0,$$

for $X \in TM$, and this follows from the bihermitian property of g. To prove 3.4, we use the commutation relation $(I_+ + I_-)I_+ = I_-(I_+ + I_-)$, which also holds for U, and the fact that g is Hermitian:

$$FI_+ = -2gUI_+ = -2gI_-U = 2I_-^* gU = -I_-^* F.$$

To prove the closure of F, we first express dF in terms of the Levi-Civita connection ∇, and then use equation 3.3 to express it in terms of the metric, U, and the complex structures

$$\begin{aligned} dF(X,Y,Z) &= (\nabla_X F)(Y,Z) + c.p. \\ &= -2g(\nabla_X(U)Y, Z) + c.p. \\ &= -2g((\nabla_X(I_+ + I_-))UY, UZ) + c.p., \end{aligned}$$

where *c.p.* means that we take cyclic permutations. Now using equation 3.2, and the type decompositions for $\omega_\pm$ and H, we find that

$$\begin{aligned} -2g((\nabla_X I_\pm)Y, Z) &= \mp(H(I_\pm X, I_\pm Y, I_\pm Z) - H(I_\pm X, Y, Z)) \\ &= \mp(H(X, Y, I_\pm Z) + H(X, I_\pm Y, Z)), \end{aligned}$$

so that

$$dF(X,Y,Z) = -H(X, UY, I_+UZ) - H(X, I_+UY, UZ) + H(X, UY, I_-UZ) + H(X, I_-UY, UZ) + c.p.,$$

and using the identity $X = I_+UX + I_-UX$, we get

$$\begin{aligned} &H(I_+UX, I_-UY, UZ) + H(I_-UX, I_-UY, UZ) + H(I_+UX, UY, I_-UZ) + H(I_-UX, UY, I_-UZ) \\ &- H(I_+UX, I_+UY, UZ) - H(I_-UX, I_+UY, UZ) - H(I_+UX, UY, I_+UZ) - H(I_-UX, UY, I_+UZ) + c.p.. \end{aligned}$$

Now focus on the mixed terms involving both complex structures. Using the cyclic permutation, we can trade $H(I_-UX, I_+UY, UZ)$ for $H(I_-UZ, I_+UX, UY)$, which cancels $H(I_+UX, UY, I_-UZ)$, and we can trade $H(I_-UX, UY, I_+UZ)$ for $H(I_-UY, UZ, I_+UX)$, which cancels $H(I_+UX, I_-UY, UZ)$. This leaves us with only the terms involving a single complex structure. Summing up all the terms involving only I_-, and using the type decomposition of H, we get

$$2(H(I_-UX, I_-UY, UZ) + H(I_-UX, UY, I_-UZ) + H(UX, I_-UY, I_-UZ)) = 2H(UX, UY, UZ),$$

and doing the same for the terms involving I_+, we get the same result, but with opposite sign. Therefore, we conclude that $dF = 0$. □

All of the tensors involved in a GK structure can in fact be expressed, using the two complex structures, in terms of the symplectic form F.

Lemma 3.1.4. *The Hermitian forms $\omega_\pm = gI_\pm$ coincide with the $(1,1)$ components of the symplectic* *form F, relative to the complex structures $I_\pm$:*

$$\omega_\pm = F^{(1,1)_\pm},$$

and therefore, we obtain the following expression for the metric:

$$g = -F^{(1,1)_\pm} I_\pm.$$

Furthermore, $d^c_\pm \omega_\pm = \frac{1}{2} d^c_\pm F$, so that $H = \frac{1}{2} d^c_+ F$.

Proof. Using expression 3.3 for the metric, and identity 3.4, we see that

$$\omega_\pm = gI_\pm = -\frac{1}{2}F(I_+ + I_-)I_\pm = \frac{1}{2}(F + I^*_\pm F I_\pm) = F^{(1,1)_\pm}.$$

Next, using the fact that F is closed, and $\omega_\pm$ is of type $(1,1)$, we get

$$d^c_\pm \omega_\pm = (I^a_\pm)^{-1} d\omega_\pm = \frac{1}{2}(I^a_\pm)^{-1} dI^a_\pm F = \frac{1}{2} d^c_\pm F.$$

□

It turns out that the symplectic type condition on a GK structure imposes a strong restriction on the cohomology class of the WZW form H, forcing it to be trivial. Indeed, there is a canonical trivialization, given by

$$b = -\frac{1}{2}F(I_+ - I_-), \tag{3.5}$$

and which is known as the *B-field potential*.

Proposition 3.1.5. *The B-field b is given by the following expression involving F and either I_+ or I_-*

$$b = \mp F^{(2,0)+(0,2)_\pm} I_\pm.$$

Therefore, b is of type $(2,0)+(0,2)$ with respect to both complex structures. Furthermore, $db = -\frac{1}{2} d^c_+ F$, so that $H = -db$.

Proof. Using identity 3.4, the definition of b can be rewritten as

$$b = \mp\frac{1}{2}(FI_\pm + I^*_\pm F) = \mp(iF^{(2,0)_\pm} - iF^{(0,2)_\pm}) = \mp F^{(2,0)+(0,2)_\pm} I_\pm.$$

Note also that $b = \mp\frac{1}{2}(FI_\pm + I^*_\pm F) = \mp\frac{1}{2} I^d_\pm F$, and hence

$$db = \mp\frac{1}{2} dI^d_\pm F = \mp\frac{1}{2}[d, I^d_\pm]F = \mp\frac{1}{2} d^c_\pm F,$$

where we have used the fact that F is closed. □

3.1.1 Hitchin Poisson structure

A surprising result of Hitchin [56] is that a GK structure (g, I_+, I_-) gives rise to two Poisson structures $\sigma_\pm$, which are holomorphic for the respective complex structure $I_\pm$, and have the same imaginary part.

They are given by

$$\sigma_{\pm} = -\frac{1}{4}(I_{\pm}Q + iQ),$$

for Q defined by the following map from the cotangent to the tangent bundle

$$Q = \frac{1}{2}[I_-, I_+]g^{-1}.$$

Evidently, these Poisson structures measure the failure of commutativity between I_+ and I_-. In the example of a Kähler structure, therefore $Q = 0$. Furthermore, the symplectic leaves of the two Poisson structures coincide, with tangent space given by the image of the commutator $[I_-, I_+]$.

We will presently prove that $\sigma_{\pm}$ are indeed holomorphic Poisson in the case of GK geometry of symplectic type. In order to do this, we use the following identity

$$[I_-, I_+] = (I_- - I_+)(I_+ + I_-),$$

which allows us to express Q in terms of F as

$$Q = \frac{1}{2}(I_- - I_+)U^{-1}g^{-1} = (I_+ - I_-)F^{-1},$$

which we can write as the following compatibility condition

$$I_+ - I_- = QF. \tag{3.6}$$

Using identity 3.4, we can then write $\sigma_{\pm}$ in terms of F as follows

$$\begin{aligned}
\sigma_{\pm} &= -\frac{1}{4}(I_{\pm}(I_+ - I_-)F^{-1} + i(I_+ - I_-)F^{-1}) \\
&= \pm\frac{1}{4}(F^{-1} - I_{\pm}F^{-1}I_{\pm}^* - iI_{\pm}F^{-1} - iF^{-1}I_{\pm}^*) \\
&= \pm\frac{1}{4}(1 - iI_{\pm})F^{-1}(1 - iI_{\pm}^*) \\
&= \pm(F^{-1})^{(2,0)_{\pm}},
\end{aligned}$$

Hence, the Hitchin Poisson structures $\sigma_{\pm}$ are given by the $(2,0)$ component, with respect to $I_{\pm}$, of the Poisson structure associated to F.

Proposition 3.1.6. *[56, Proposition 5] The bivectors $\sigma_{\pm}$ define holomorphic Poisson structures, with respect to $I_{\pm}$ respectively.*

Proof. We will only prove the result for σ_+, as the other case works identically. We are also working under the assumption that our GK structure is of symplectic type. Since σ_+ is given by the $(2,0)$ component of F^{-1}, it follows that for I_+-holomorphic functions f, h, we have

$$\{f, h\}_{\sigma_+} = \{f, h\}_F.$$

It therefore suffices to show that the bracket induced by F preserves the I_+-holomorphic functions. If this is satisfied, then the Jacobi identity for σ_+ follows immediately from the fact that F is symplectic.

So let f and h be holomorphic functions. Using equation 3.3, we can rewrite their bracket as follows:

$$\{f,h\} = F^{-1}(df,dh) = \frac{1}{2}g^{-1}((I_+ + I_-)df,dh) = \frac{1}{2}g^{-1}(idf + I_-df,dh) = \frac{1}{2}g^{-1}(I_-df,dh),$$

where the last equality follows from the fact that g is of type $(1,1)$. We will show that this expression is holomorphic. Differentiating with respect to a $(0,1)$ vector field X, we get

$$X(g^{-1}(I_-df,dh)) = g^{-1}(\nabla^+_X(I_-)df,dh) + g^{-1}(I_-\nabla^+_X df,dh) + g^{-1}(I_-df,\nabla^+_X dh), \tag{3.7}$$

where ∇^+ is the Bismut connection of (g,I_+). Using the relation, $\nabla^+ = \nabla^- - g^{-1}H$, between the Bismut connections of $(g,I_\pm)$, we find that

$$g^{-1}(\nabla^+_X(I_-)df,dh) = H(X,I_-g^{-1}(df),g^{-1}(dh)) + H(X,g^{-1}(df),I_-g^{-1}(dh)).$$

To evaluate the last two terms of the sum 3.7, we write the de Rham differential in terms of the Bismut connection

$$0 = d(df)(X,Y) = (\nabla^+_X df)(Y) - (\nabla^+_Y df)(X) + H(X,g^{-1}df,Y),$$

where Y is some test vector field, and the last term comes from the torsion $-g^{-1}H$. The connection ∇^+ preserves the type decomposition for I_+, and hence the second term vanishes, and we are left with the following identity

$$\nabla^+_X df = -H(X,g^{-1}df).$$

Plugging this into the last two terms of equation 3.7, we get

$$-H(X,g^{-1}(df),I_-g^{-1}(dh)) + H(X,g^{-1}(dh),I_-g^{-1}(df)),$$

which cancels the first term. □

3.1.2 Degenerate GK structures of Symplectic type

The upshot of the discussion so far is that a GK structure of symplectic type, (g,I_+,I_-), gives rise to a symplectic form F, and a pair of holomorphic Poisson structures $\sigma_\pm$, which satisfy equations 3.4 and 3.6, and we can recover the GK structure from this data via equation 3.3. However, it is clear that data such as $(F,\sigma_\pm)$, satisfying equations 3.4 and 3.6, may fail to define a GK structure, if the tensor defined by equation 3.3 fails to be positive definite. We will find it useful to relax this positivity condition, and consider *degenerate GK structures.*

Definition 3.1.7 (Degenerate Generalized Kähler Structures)**.** A *degenerate* GK structure of symplectic type consists of the data (I_+,I_-,Q,F) of a closed 2-form $F \in \Omega^{2,cl}(M,\mathbb{R})$, a real Poisson structure Q, and two complex structures $I_\pm$, such that Q is the common imaginary part of holomorphic Poisson structures for $I_\pm$

$$\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ),$$

and such that the following two equations are satisfied

$$I_+ - I_- = QF, \tag{3.8}$$

$$FI_+ + I_-^* F = 0. \tag{3.9}$$

Remark 3.1.8. This way of encoding a GK structure via equations 3.8 and 3.9 was first considered in [46]. This system of equations was also studied in [62].

Example 3.1.9. Associated to any holomorphic Poisson structure $(I, \sigma = -\frac{1}{4}(IQ + iQ))$, there is a canonical degenerate GK structure given by $(I, I, Q, 0)$. The associated symmetric tensor is $g = 0$, and so this does not define a GK structure. ■

Given a degenerate GK structure (I_+, I_-, Q, F) it is possible to define a symmetric Hermitian tensor g via equation 3.3, and a 2-form b via equation 3.5. It is then easy to check that lemma 3.1.4, and proposition 3.1.5 remain valid in this setting, and hence the associated Hermitian forms $\omega_\pm$ satisfy the integrability condition 3.1. In other words, the data (g, I_+, I_-) satisfy all the conditions of GK geometry, except for the positive definiteness of g. We say that the form F is *positive* if g is a positive-definite Riemannian metric. Note that this is really a condition on (I_+, I_-, F): in particular, F must be symplectic, and $(I_+ + I_-)$ must be invertible.

Summarizing the discussion, we have proved the following equivalence.

Theorem 3.1.10. *[46, Theorem 6.2] There is a bijection between Generalized Kähler structures of symplectic type (g, I_+, I_-) and degenerate GK structures of symplectic type (I_+, I_-, Q, F) such that the 2-form F is positive.*

In this thesis, we will actually work with the wider class of degenerate GK structures, applying the condition of positivity of F when we want to make contact with GK geometry.

3.1.3 Generalized Complex geometry

In this section we recall the observations made in [46] about the meaning of equations 3.8 and 3.9 in terms of generalized complex geometry. To this end, we introduce some basics of the theory (see [44, 47] for a general treatment).

In its most basic form, generalized complex geometry involves doing geometry on the generalized tangent bundle $TM \oplus T^*M$, which is equipped with a natural non-degenerate split-signature symmetric pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)),$$

and the *Courant bracket*

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X(\eta) - \iota_Y(d\xi),$$

where $X + \xi$ and $Y + \eta$ are sections of $TM \oplus T^*M$. This bracket satisfies the Jacobi identity, but is skew-symmetric only up to a term involving the pairing. Closed 2-forms B provide natural symmetries of these structures, via the following automorphism

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

An almost generalized complex (GC) structure $\mathbb{J}$ is an orthogonal endomorphism of the generalized tangent bundle, which squares to -1. When we pass to the complexification, this endomorphism has $\pm i$ eigenbundles, and these are *Lagrangian* for the symmetric pairing, as a result of $\mathbb{J}$ being orthogonal. A generalized complex structure $\mathbb{J}$ is then defined to be an *integrable* almost GC structure, which means that its $+i$ eigenbundle is closed under the Courant bracket. Basic examples of GC structures are given by complex structures I, and symplectic structures ω, for which the endomorphisms are given by

$$\begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Consider an almost GC structure, $\mathbb{J}_\sigma$, whose matrix is upper triangular. It is routine to check that it must have the form

$$\mathbb{J}_\sigma = \begin{pmatrix} -I & Q \\ 0 & I^* \end{pmatrix},$$

for I an almost complex structure, and Q a bivector field of type $(2,0)+(0,2)$. The $+i$ eigenbundle of such an almost GC structure is given by

$$L_\sigma = \{X + \sigma(\zeta) + \zeta \mid X \in T^{0,1}M,\ \zeta \in T^*_{1,0}M\},$$

where $\sigma = -\frac{1}{4}(IQ + iQ)$. Analyzing the condition that L_σ is closed under the Courant bracket, we see that $\mathbb{J}_\sigma$ defines a GC structure if and only if I defines a complex structure, and σ defines a holomorphic Poisson bracket.

Equations 3.8 and 3.9 can be repackaged in terms of the GC structures associated to $(I_\pm, \sigma_\pm)$ to give

$$\begin{pmatrix} -I_+ & Q \\ 0 & I_+^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} -I_- & Q \\ 0 & I_-^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix},$$

or more compactly, $\mathbb{J}_{\sigma_+} = e^F \mathbb{J}_{\sigma_-} e^{-F}$. Now since the closed 2-form F defines a symmetry of $TM \oplus T^*M$, with its pairing and bracket, this relation implies the following non-trivial consequence: the integrability of one of the pair $(I_\pm, \sigma_\pm)$ follows from the other. This means that it suffices to include a single holomorphic Poisson structure in the definition of a degenerate GK structure. More precisely, the defining equations 3.8 and 3.9 of a degenerate GK structure can be combined into a single equation, involving only I_-, F, and Q:

$$FI_- + I_-^* F + FQF = 0. \tag{3.10}$$

The other complex structure can then be defined to be $I_+ = I_- + QF$, and the integrability of the pair (I_+, σ_+) follows from the integrability of (I_-, σ_-) and the closure of F. We therefore obtain the following result.

Proposition 3.1.11. *Let (I_-, Q) define a holomorphic Poisson structure, and let F be a real closed 2-form, such that equation 3.10 is satisfied. Then with $I_+ = I_- + QF$, the data (I_+, I_-, Q, F) defines a degenerate GK structure of symplectic type.*

Chapter 4

Branes in Morita equivalences

In this chapter, we present the main reformulation of generalized Kähler geometry of symplectic type, and thereby explain the inherent degrees of freedom in GK geometry. More precisely, we answer the following two questions.

1. What is the holomorphic structure underlying a Generalized Kähler manifold of symplectic type?
2. What additional data is needed to specify the Riemannian metric?

4.1 Holomorphic symplectic Morita equivalences

In this section, we provide an answer to the first question, explaining that the underlying holomorphic structure of a (degenerate) GK structure of symplectic type consists of a holomorphic symplectic Morita equivalence between the two underlying holomorphic Poisson structures.

Let (I_+, I_-, Q, F) be a degenerate GK structure of symplectic type on a manifold M, and let $\sigma_\pm$ denote the two holomorphic Poisson structures. We also denote the two complex manifolds by $X_\pm = (M, I_\pm)$. Our goal is to construct a holomorphic symplectic Morita equivalence between (X_-, σ_-) and (X_+, σ_+). Following [4], we first prove, as a preliminary step, that the integration of σ_+ can be constructed from the integration of σ_-, by choosing a closed 2-form on the groupoid which satisfies equation 3.10. Indeed, note that since these two Poisson structures have the same imaginary parts, it follows that their integrations have the same underlying 'imaginary' symplectic groupoid $(\mathcal{G}, \omega)$, which is an integration of Q.

Proposition 4.1.1. *[4, Proposition 6.3] Let $\Sigma(X_-)$ be the holomorphic symplectic groupoid of σ_-, with complex structure I_-, and holomorphic symplectic form $\Omega_- = B_- + i\omega$, and let $(\mathcal{G}, \omega)$ denote the underlying 'imaginary' smooth symplectic groupoid integrating (M, Q). The holomorphic symplectic groupoid of σ_+, denoted $\Sigma(X_+)$, is then given by $\mathcal{G}$, with complex structure $I_+ = I_- + \omega^{-1}C$, and holomorphic symplectic form $\Omega_+ = \Omega_- + C$, where $C = t^*F - s^*F$. Furthermore, the data $(I_+, I_-, \omega^{-1}, C)$ defines a multiplicative degenerate GK structure of symplectic type on the groupoid $\mathcal{G}$.*

Proof. The strategy of the proof is to first show that I_-, ω, and C satisfy equation 3.10, which by Proposition 3.1.11 implies that Ω_+ and I_+ define a holomorphic symplectic structure on $\mathcal{G}$. We then have to show that these structures are multiplicative, and hence define a holomorphic symplectic groupoid, and that the induced Poisson structure on the base is σ_+.

Let's start by setting notation, and making a few observations. We denote the pullback of a form, such as the pullback of F via t, by $t^*(F)$. This is to distinguish it from t^*F, which is the composition of the bundle map induced by F, and the pullback $t^* : T^*M \to T^*\mathcal{G}$. The bundle map associated to $t^*(F)$ is therefore given by t^*Ft_*, where $t_* : T\mathcal{G} \to TM$. The source and target maps are holomorphic, and hence satisfy $t_*I_- = I_-t_*$, and $s_*I_- = I_-s_*$. And since the underlying imaginary part of $\Sigma(X_-)$ is the symplectic groupoid $(\mathcal{G}, \omega)$ integrating (M, Q), we have that

$$t_*\omega^{-1}t^* = Q, \qquad s_*\omega^{-1}s^* = -Q, \qquad t_*\omega^{-1}s^* = s_*\omega^{-1}t^* = 0,$$

reflecting the fact that t is Poisson, s is anti-Poisson, and the source and target fibres are symplectic orthogonal, respectively.

We are now ready to check equation 3.10 for I_-, ω, and C. Indeed, using the above identities, and

$$FI_- + I_-^*F + FQF = 0$$

on M, we find that

$$\begin{aligned} CI_- + I_-^*C + C\omega_-^{-1}C &= (t^*Ft_* - s^*Fs_*)I_- + I_-^*(t^*Ft_* - s^*Fs_*) \\ &\qquad + (t^*Ft_* - s^*Fs_*)\omega^{-1}(t^*Ft_* - s^*Fs_*) \\ &= t^*(FI_- + I_-^*F + FQF)t_* - s^*(FI_- + I_-^*F + FQF)s_* \\ &\qquad - t^*F(t_*\omega^{-1}s^*)Fs_* - s^*F(s_*\omega^{-1}t^*)Ft_* \\ &= 0. \end{aligned}$$

This shows that (Ω_+, I_+) is holomorphic symplectic by Proposition 3.1.11.

Next, we check that source and target maps are holomorphic, with respect to the complex structure I_+ on $\mathcal{G}$, and $I_+ = I_- + QF$ on M:

$$t_*I_+ = t_*I_- + t_*\omega^{-1}(t^*Ft_* - s^*Fs_*) = I_-t_* + (t_*\omega^{-1}t^*)Ft_* - (t_*\omega^{-1}s^*)Fs_* = (I_- + QF)t_* = I_+t_*,$$

the case with s being similar.

A straightforward calculation on the space $\mathcal{G}^{(2)}$ of composable arrows shows that $C = t^*(F) - s^*(F)$ is multiplicative. Therefore $\Omega_+ = \Omega_- + C$, a sum of multiplicative forms, is also multiplicative. Since I_+ is determined by Ω_+, in the sense that $I_+ = (\mathrm{Im}\Omega_+)^{-1}(\mathrm{Re}\Omega_+)$, it then immediately follows that it is also multiplicative. For example, showing that the multiplication map is holomorphic is equivalent to showing that its graph is a complex submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$, and this is implied by the fact that the graph is an Ω_+-Lagrangian submanifold.

Finally, we need to show that $t_*(\Omega_+^{-1}) = \sigma_+$, and $s^*(\Omega_+^{-1}) = -\sigma_+$. But since σ_+ is determined by Q and I_+, this follows immediately from what we have already shown. □

Let us record the following simple observation.

Lemma 4.1.2. *Let $\mathcal{R} \rightrightarrows M \coprod N$ be a Lie groupoid, whose base is the disjoint union of M and N. Then $\mathcal{G} = s^{-1}(M) \cap t^{-1}(M) \rightrightarrows M$ and $\mathcal{H} = s^{-1}(N) \cap t^{-1}(N) \rightrightarrows N$ are Lie groupoids over M and N, respectively, $P = t^{-1}(M) \cap s^{-1}(N)$ is a Morita equivalence from $\mathcal{H}$ to $\mathcal{G}$, and $Q = t^{-1}(N) \cap s^{-1}(M)$ is the inverse Morita equivalence from $\mathcal{G}$ to $\mathcal{H}$. Furthermore, all Morita equivalences between $\mathcal{G}$ and $\mathcal{H}$*

arise in this way. This result remains true if we work with symplectic groupoids and symplectic Morita equivalences.

We are now ready to construct the Morita equivalence between σ_- and σ_+. Start with the symplectic groupoid $(\Sigma(X_-), \Omega_-) \rightrightarrows X_-$ integrating σ_-. Now construct the symplectic groupoid $(\mathcal{R}, \Omega)$ over the disjoint union $(Y, \sigma) = (X_-^{(2)}, \sigma_-) \coprod (X_-^{(1)}, \sigma_-)$, whose space of arrows, $\mathcal{R}$, is defined by setting

$$(t^{-1}(X_-^{(i)}) \cap s^{-1}(X_-^{(j)}), \Omega) = (\Sigma(X_-), \Omega_-),$$

and whose multiplication is defined in the obvious way. Hence, this is the groupoid which corresponds, via Lemma 4.1.2, to the trivial Morita self-equivalence of $\Sigma(X_-)$. Now consider the closed 2-form $\tilde{F}$ on Y, defined to be F on $X_-^{(2)}$ and 0 on $X_-^{(1)}$. This defines a degenerate GK structure, where the new holomorphic Poisson structure is $(\tilde{Y}, \tilde{\sigma}) = (X_+, \sigma_+) \coprod (X_-, \sigma_-)$. Hence by Proposition 4.1.1, $(\mathcal{R}, \Omega + t^*\tilde{F} - s^*\tilde{F})$ defines a holomorphic symplectic groupoid over $\tilde{Y}$, and as one can check $t^{-1}(X_+) \cap s^{-1}(X_+) = (\Sigma(X_+), \Omega_+)$, and $t^{-1}(X_-) \cap s^{-1}(X_-) = (\Sigma(X_-), \Omega_-)$. Then, by Lemma 4.1.2, $t^{-1}(X_+) \cap s^{-1}(X_-)$ provides a holomorphic symplectic Morita equivalence between these two subgroupoids. We have therefore proven:

Proposition 4.1.3. *[4, Proposition 6.4] Let (I_+, I_-, Q, F) be a degenerate GK structure of symplectic type, and let $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$ denote the corresponding holomorphic Poisson structures on the complex manifolds $X_\pm = (M, I_\pm)$. Then (X_+, σ_+) and (X_-, σ_-) are holomorphically symplectically Morita equivalent in a canonical way.*

Unpacking the construction of the Morita equivalence, we see that it is given by

$$\begin{tikzcd} & (Z,\Omega) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ (X_+,\sigma_+) & & (X_-,\sigma_-) \end{tikzcd}$$

where

$$(Z, \Omega) = (\Sigma(X_-), \Omega_- + t^*F), \tag{4.1}$$

with $\pi_+ = t$, and $\pi_- = s$, the target and source maps of $\Sigma(X_-)$, respectively. Recall that since Ω is a holomorphic symplectic form, it uniquely determines its own complex structure, and hence specifying this form is enough. This Morita equivalence represents the underlying holomorphic structure of the degenerate GK structure of symplectic type.

4.2 Generalized Kähler metrics as brane bisections

So far, we have shown how to extract an underlying holomorphic symplectic Morita equivalence (Z, Ω) from the data of a (degenerate) GK structure of symplectic type (I_+, I_-, Q, F). In this section, we consider the question of how to represent the metric. In order to recover the GK structure from Z, we need some real (i.e. non-holomorphic) data. To see how this arises, observe that since Z was obtained by deforming the holomorphic symplectic groupoid $\Sigma(X_-)$ by t^*F, it contains a distinguished submanifold $\mathcal{L}$ coming from the identity bisection. This submanifold is neither holomorphic nor Lagrangian in (Z, Ω). It is, however, characterized by the following two properties:

1. The submanifold $\mathcal{L} \subset Z$ is a smooth bisection. This defines a diffeomorphism between the underlying smooth manifolds of X_+ and X_-, which is required because a GK structure involves two complex structures living on the same real manifold.

2. The bisection $\mathcal{L}$ is Lagrangian with respect to $\omega = \mathrm{Im}(\Omega)$: from the point of view of generalized complex geometry, this is known as a *brane* in (Z, Ω). Notice that equation 4.1 implies that $\Omega|_{\mathcal{L}} = F$, so that we recover the real closed 2-form F.

We record the above properties satisfied by $\mathcal{L}$ in the following definition.

Definition 4.2.1 (Brane bisection)**.** A *brane bisection* in the Morita equivalence (Z, Ω) is a smooth submanifold of Z which is Lagrangian for $\mathrm{Im}(\Omega)$ and which is a section of both π_- and π_+.

Theorem 4.2.2. *[6, Theorem 4.3] A degenerate GK structure of symplectic type (I_+, I_-, Q, F) is equivalent to a holomorphic symplectic Morita equivalence with brane bisection $(Z, \Omega, \mathcal{L})$ between the holomorphic Poisson structures $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$.*

Proof. One direction of the theorem follows from Proposition 4.1.3 and the above remarks. For the converse direction let us start with a holomorphic symplectic Morita equivalence with brane bisection $(Z, \Omega = B + i\omega, \mathcal{L})$ and construct a degenerate GK structure. Such a Morita equivalence goes between two holomorphic Poisson manifolds $(X_\pm, \sigma_\pm = -\frac{1}{4}(I_\pm Q_\pm + iQ_\pm))$. Now observe that (Z, ω) defines a real smooth symplectic Morita equivalence between the real Poisson structures (X_+, Q_+) and (X_-, Q_-). The fact that $\mathcal{L}$ is a brane bisection means precisely that it defines a Lagrangian bisection in this Morita equivalence, and therefore that it defines a Poisson diffeomorphism between the two Poisson structures. The upshot of this is that when we use the bisection to identify $X_- = \mathcal{L} = X_+$, then $Q_- = Q_+$. We denote this real Poisson structure by Q and the underlying smooth manifold by M.

Now that we have made these identifications (Z, ω) defines a self-Morita equivalence of (M, Q) and $\mathcal{L}$ is a Lagrangian bisection inducing the identity diffeomorphism on M. The symplectic groupoid $(\Sigma(X_-), \Omega_- = B_- + i\omega_-)$ of (X_-, σ_-) acts principally on (Z, Ω), and so the brane $\mathcal{L}$ induces the following isomorphism of smooth symplectic Morita equivalences

$$\phi : (\Sigma(X_-), \omega_-) \to (Z, \omega), \ g \mapsto \lambda(t(g)) * g,$$

where $\lambda : M \to Z$ is the section of π_- induced by $\mathcal{L}$. Using ϕ to compare the real parts of Ω_- and Ω we see that

$$\phi^*(B) = B_- + t^*(B|_{\mathcal{L}}). \tag{4.2}$$

Let $F := B|_{\mathcal{L}}$. Then we see that under this isomorphism

$$\phi^*(\Omega) = \Omega_- + t^*(F).$$

On the manifold M, we now have the data of the two complex structures $I_\pm$ coming from the spaces $X_\pm$, the real Poisson structure Q, which is the common imaginary part of the holomorphic Poisson structures $\sigma_\pm$, and a closed 2-form F. We then show that (I_+, I_-, Q, F) satisfy equations 3.8 and 3.9. For simplicity, we use ϕ to identity (Z, ω) with $(\Sigma(X_-), \omega_-)$. On this space, we have complex structures I and I_- coming from Z and $\Sigma(X_-)$, respectively. Note that the complex structure I satisfies

$$t_* I = I_+ t_*, \qquad s_* I = I_- s_*.$$

In order to verify equations 3.8 and 3.9, we show that they hold for $(I, I_-, \omega^{-1}, t^*(F))$ on $\Sigma(X_-)$ and therefore on M by push-forward. First we note that since $B = \omega I$ and $B_- = \omega I_-$, we have from equation 4.2 that $\omega(I - I_-) = t^*(F)$, which we can rearrange to equation 3.8:

$$I - I_- = \omega^{-1} t^*(F).$$

Applying the target projection we get

$$(I_+ - I_-)t_* = t_* \omega^{-1} t^* F t_* = QF t_*,$$

which implies equation 3.8 on M since t_* is surjective. Note that we write $t^*(F)$ for the pullback of the 2-form, which gives rise to the bundle map $t^* F t_* : T \to T^*$. Equation 3.9 on the groupoid follows from a direct computation:

$$I^* t^*(F) + t^*(F) I_- = I^* \omega(I - I_-) + \omega(I - I_-) I_- = 0,$$

where we have used the identities $t^*(F) = \omega(I - I_-)$ and $I^*\omega = \omega I$. But this implies the corresponding equation on M since

$$t^*(I_+^* F + F I_-) = I^* t^*(F) + t^*(F) I_- = 0,$$

and t^* is injective on differential forms. This establishes equations 3.8 and 3.9 on M. It is clear that the two constructions outlined are inverse to each other, so we get the desired equivalence. □

Remark 4.2.3. In the statement of Theorem 4.2.2, the Morita equivalence is assumed to go between Poisson manifolds, and hence by Theorem 2.2.23, it is a Morita equivalence between the source simply connected integrations. However, nowhere in the proof of this theorem is the integration required to be source simply connected, and in fact, we can omit this assumption. Hence, fixing an integration $\mathcal{G}_-$ of σ_-, there is a bijection between degenerate GK structures of symplectic type, and Morita equivalences with brane bisection between $\mathcal{G}_-$ and some integration of σ_+.

Definition 4.2.4. A brane bisection $\mathcal{L}$ in the Morita equivalence (Z, Ω) is *positive* if $F = \Omega|_{\mathcal{L}}$ is *positive.*

Corollary 4.2.5. *There is a bijection between generalized Kähler structures of symplectic type, and holomorphic symplectic Morita equivalences with positive brane bisection.*

Theorem 4.2.2 answers the main questions raised at the beginning of this chapter, identifying the Morita equivalence (Z, Ω) as the underlying holomorphic data of a GK manifold of symplectic type, and showing how the additional data of a brane bisection $\mathcal{L}$ specifies the GK metric.

4.2.1 Induced metric

We now discuss how the metric and B-field potential of the GK structure arise from the holomorphic symplectic Morita equivalence with brane bisection $(Z, \Omega, \mathcal{L})$. From the equivalence established in Theorem 4.2.2, we know that a Morita equivalence with brane bisection gives rise to a degenerate GK structure, and from this we can extract the data of a (possibly degenerate) metric and B-field potential. In this section, we detail how to determine the metric and B-field directly from the geometry of the Morita equivalence.

Given a real 2-form F and a complex structure I, the tensor FI decomposes into symmetric and anti-symmetric parts: $FI = S + A$, where

$$S = F^{(1,1)}I, \qquad A = F^{(2,0)+(0,2)}I.$$

Given a Morita equivalence with brane bisection $(Z, \Omega, \mathcal{L})$, we have two induced complex structures $I_\pm$ on the brane, as well as the real 2-form $F = \Omega|_{\mathcal{L}}$. Therefore the tensors $FI_\pm$ give rise to symmetric tensors $S_\pm = F^{(1,1)_\pm}I_\pm$ and anti-symmetric tensors $A_\pm = F^{(2,0)+(0,2)_\pm}I_\pm$. From the theory of GK structures we know to expect that $S_+ = S_-$ and $A_+ = -A_-$. Indeed, from Lemma 3.1.4 we know that $g = -S_\pm$, and from Lemma 3.1.5 we know that $b = \mp A_\pm$. Here we explain these facts directly.

Proposition 4.2.6. *Let $(Z, \Omega, \mathcal{L})$ be a holomorphic symplectic Morita equivalence with brane bisection, let $I_\pm$ denote the complex structures induced on the brane, and let $F = \Omega|_{\mathcal{L}}$ denote the real 2-form on the brane obtained by pulling back the symplectic form. If $S_\pm = F^{(1,1)_\pm}I_\pm$ and $A_\pm = F^{(2,0)+(0,2)_\pm}I_\pm$, then*

$$S_+ = S_- \quad \textit{and} \quad A_+ = -A_-.$$

Proof. Let $\Omega = B + i\omega$ be the holomorphic symplectic form on Z, so that $F = B|_{\mathcal{L}}$. Writing $A_\pm$ and $S_\pm$ as the (anti-)symmetrizations of $FI_\pm$, we have

$$2S_\pm(u,v) = B(I_\pm u, v) + B(I_\pm v, u), \qquad 2A_\pm(u,v) = B(I_\pm u, v) - B(I_\pm v, u),$$

for $u, v \in T\mathcal{L}$. Using the two direct sum decompositions of TZ along $\mathcal{L}$ determined by the fibres of the projections $\pi_\pm$,

$$TZ|_{\mathcal{L}} = T\mathcal{L} \oplus K_\pm, \qquad K_\pm = \ker(d\pi_\pm),$$

we can write the following expressions for $I_\pm$:

$$Iu = I_\pm u + k_\pm u, \qquad I_\pm u \in T\mathcal{L}, \ k_\pm u \in K_\pm,$$

for $u \in T\mathcal{L}$. Then using the fact that $K_\pm$ are ω-symplectic orthogonal, $T\mathcal{L}$ is ω-Langrangian, and $\omega I = I^*\omega$, we get the following identity

$$0 = \omega(k_+u, k_-v) = \omega(Iu - I_+u, Iv - I_-v) = \omega(Iv, I_+u) - \omega(Iu, I_-v),$$

for $u, v \in T\mathcal{L}$. Using the relation $B = \omega I$, this gives the following identity:

$$B(I_+u, v) = B(I_-v, u),$$

which implies the two identities above for $S_\pm$ and $A_\pm$. □

The upshot of the present discussion is as follows: when we pullback the holomorphic $(2,0)$-form Ω along a brane we get a real 2-form F which is no longer $(2,0)$ with respect to either of the induced complex structures $I_\pm$. As such, it gives rise to symmetric and anti-symmetric tensors on the brane. The fact that F was obtained by pulling back the symplectic form of a holomorphic Morita equivalence implies that we end up with a single symmetric and anti-symmetric tensor; these give rise to the metric and B-field potential of the GK structure. The positivity or non-degeneracy of the metric arises from the configuration of the brane in the Morita equivalence. For example, g is non-degenerate if and only if $I(T\mathcal{L}) \cap K = 0 = I(T\mathcal{L}) \cap T\mathcal{L}$, where K is the average of $K_\pm$ with respect to $T\mathcal{L}$. Indeed, the condition $I(T\mathcal{L}) \cap T\mathcal{L} = 0$ is equivalent to invertibility of F and $I(T\mathcal{L}) \cap K = 0$ is equivalent to invertibility of $I_+ + I_-$. Recall that these two conditions imply that we get a GK structure with non-degenerate but possibly indefinite metric.

4.3 Examples

Lets provide a few examples of (degenerate) generalized Kähler structures of symplectic type and their corresponding holomorphic symplectic Morita equivalences with brane bisection.

Example 4.3.1 (Holomorphic Poisson structure). Let (M, I, σ) be a Poisson manifold, and let $Q = -4\mathrm{Im}(\sigma)$. As in Example 3.1.9, the associated degenerate GK structure is given by the data $(I, I, Q, 0)$. The associated Morita equivalence with brane bisection is given by the holomorphic symplectic groupoid $(\Sigma(X), \Omega)$ integrating σ, viewed as the trivial Morita self-equivalence, with brane bisection given by the identity bisection ϵ. ■

Example 4.3.2 (Kähler Manifold). Lets revisit the construction considered in Section 1.2. Recall that a Kähler manifold (M, I, g, ω) defines a generalized Kähler structure of symplectic type for which $I_\pm = I$, $Q = 0$, and $F = \omega$. Therefore the Morita equivalence is given by the holomorphic cotangent bundle of $X = (M, I)$ twisted by ω:

$$(Z, \Omega) = (T^*X, \Omega_0 + \pi^*\omega),$$

where $\pi : T^*X \to X$ is the projection and Ω_0 is the canonical symplectic form. The brane bisection $\mathcal{L}$ is given by the zero section viewed inside of Z. As discussed in Section 1.2, (Z, Ω) is an affine bundle modelled on the holomorphic cotangent bundle of X, and the brane bisection is both symplectic with respect to $\mathrm{Re}(\Omega)$ and Lagrangian with respect to $\mathrm{Im}(\Omega)$. Conversely, any such affine bundle $\mathcal{A} \to X$ gives a Morita equivalence and any section $\mathcal{L}$ of $\pi = \pi_+ = \pi_-$ which is symplectic for $\mathrm{Re}(\Omega)$ and Lagrangian for $\mathrm{Im}(\Omega)$ automatically defines a non-degenerate bilinear form on M. This is due to the fact that $K = \ker(d\pi)$ is a complex Lagrangian, which implies that $I(T\mathcal{L}) \cap \ker(d\pi) = 0$. ■

Example 4.3.3 (Hyper-Kähler Manifold). Let (M, I, J, K, g) be a hyper-Kähler structure, and let

$$(\omega_I, \omega_J, \omega_K) = (gI, gJ, gK)$$

be the corresponding triple of Kähler forms. This data defines the GK structure of symplectic type

$$(I_+, I_-, Q, F) = (I, J, \omega_K^{-1}, \omega_I + \omega_J).$$

The induced holomorphic Poisson structures are non-degenerate, with corresponding holomorphic symplectic forms

$$\Omega_+ = \omega_J + i\omega_K, \qquad \Omega_- = -\omega_I + i\omega_K.$$

Let $X_\pm = (M, I_\pm)$. Then, assuming that M is simply-connected (or using a non source-simply connected integration), we have a holomorphic Morita equivalence given by

$$(Z, \Omega) = (X_+, \Omega_+) \times (X_-, -\Omega_-),$$

with π_+ and π_- given by the projections onto the first and second factors respectively. The brane bisection $\mathcal{L}$ is then given by the diagonally embedded copy of M:

$$\mathcal{L} = \{(m, m) \mid m \in M\} \subseteq Z.$$

■

4.4 Prequantization

In this section, we describe what it means to prequantize a GK structure (I_+, I_-, Q, F). This consists of two main ingredients: a prequantization of the real 2-form F, and a 'prequantization' of the holomorphic Poisson geometry.

Definition 4.4.1. Let F be a real closed 2-form. Then a *prequantization* consists of a complex line bundle U, equipped with a connection ∇, such that the curvature $\nabla^2 = -2\pi i F$.

Recall the classical theorem of Kostant [65] and Souriau [96], which says that a closed 2-form F is prequantizable if and only if F is integral in cohomology. In the literature, it is often conventional to require U to be equipped with a Hermitian metric, such that the connection is unitary. We do not impose this condition. Note that we will abuse notation, and use U also to denote the associated principal $\mathbb{C}^*$-bundle.

Given a Poisson structure, the correct notion to consider is a multiplicative prequantization of its symplectic groupoid. This was introduced by Weinstein and Xu in [110].

Definition 4.4.2 (Multiplicative prequantization)**.** Let $(\mathcal{G} \rightrightarrows B, \Omega)$ be a symplectic groupoid. A *multiplicative prequantization* is a prequantization (U, ∇) of Ω, such that U is a multiplicative line bundle, and ∇ is a multiplicative connection. This means that there is a flat isomorphism of line bundles with connection $\mu : p_1^*(U, \nabla) \otimes p_2^*(U, \nabla) \to m^*(U, \nabla)$, over the space of composable arrows $\mathcal{G}^{(2)}$, such that μ makes the associated $\mathbb{C}^*$-principal bundle U into a Lie groupoid over B. In this case, U defines a central extension of Lie groupoids

$$\mathbb{C}^* \times B \to U \to \mathcal{G}.$$

Conditions for the existence of multiplicative prequantizations in the smooth category were studied in [110, 25, 30, 9]. A source simply connected symplectic groupoid $(\mathcal{G} \rightrightarrows B, \Omega)$ admits a multiplicative prequantization if and only if the restriction of Ω to the source-fibres has integral periods, and in this case it is unique. This condition says that for each map $\tilde{\sigma} : S^2 \to s^{-1}(x)$, for x in the base, the integral $\int_{\tilde{\sigma}} \Omega$ must be an integer. Let us call this condition *integrality* of the induced Poisson structure Q on the base of the groupoid.

It is possible to formulate this condition purely in terms of the Poisson structure Q. Recall that the Poisson structure induces a decomposition of B into immersed submanifolds $L \subseteq B$, which inherit a symplectic form ω^L. These are the symplectic leaves. Next, a map $\tilde{\sigma}$ from the 2-sphere into a source fibre is equivalent to a Lie algebroid morphism $\phi : TS^2 \to T_Q^*B$, covering the map $\sigma = t \circ \tilde{\sigma}$, which lies entirely in a symplectic leaf L. Then $\tilde{\sigma}^*(\Omega) = \sigma^*(\omega^L)$. Hence, the Poisson structure is integral if and only if for each Lie algebroid morphism $\phi : TS^2 \to T_Q^*B$, covering a base map σ which necessarily lies in a symplectic leaf L, the integral $\int_\sigma \omega^L \in \mathbb{Z}$. In particular, a Poisson structure is integral if all of its symplectic leaves have integral periods.

These structures were studied in the smooth category, however, all the above definitions make sense in the holomorphic category as well. By a holomorphic prequantization of a closed holomorphic 2-form Ω, we mean a holomorphic line bundle L, equipped with a holomorphic connection ∇, whose curvature is given by $-2\pi i\Omega$. Note that this is a slightly disturbing concept: the holomorphic line bundle L can always be equipped with a hermitian metric, and the associated Chern connection has curvature given by a closed $(1,1)$-form. Hence the $(2,0)$-form Ω is cohomologous to a $(1,1)$-form. If we are in the setting of a compact Kähler manifold, this can only happen if Ω is cohomologically trivial, and by the $\partial\bar{\partial}$-lemma, this implies that $\Omega = 0$. In particular, there is no holomorphic prequantization of a holomorphic symplectic form on a compact Kähler manifold. Nevertheless, we make the following definition.

Definition 4.4.3 (GK prequantization)**.** A prequantization of a GK structure (I_+, I_-, Q, F) consists of a holomorphic multiplicative prequantization of the holomorphic symplectic groupoid associated to the Morita equivalence of Theorem 4.2.2 via Lemma 4.1.2. More precisely, this consists of

1. multiplicative prequantizations $(U_\pm, \nabla_\pm)$ of the holomorphic symplectic groupoids $(\Sigma(X_\pm), \Omega_\pm)$ integrating the holomorphic Poisson structures $\sigma_\pm$;
2. a holomorphic prequantization (U, ∇) of the Morita equivalence (Z, Ω), which is compatible with the prequantizations $U_\pm$ in the sense that U provides a Morita equivalence between $U_\pm$, and the action maps
$$(U_+, \nabla_+) \otimes (U, \nabla) \to (U, \nabla), \qquad (U, \nabla) \otimes (U_-, \nabla_-) \to (U, \nabla),$$
are flat.

Remark 4.4.4. Based on the observations preceding this definition, it might appear that multiplicative prequantizations are very difficult to find. However, symplectic groupoids are often non-compact, and so a $(2,0)$-form can in fact be cohomologous to a $(1,1)$-form.

Theorem 4.4.5. *A degenerate GK structure of symplectic type (I_+, I_-, Q, F) is prequantizable if and only if F is integral in cohomology and the holomorphic Poisson structure σ_- is integral.*

Remark 4.4.6. For simplicity, we assume that the integration of Q is Hausdorff.

Proof. One direction is obvious: if (I_+, I_-, Q, F) is prequantizable, then by definition, the symplectic groupoid $(\Sigma(X_-), \Omega_-)$ is prequantizable, which implies that σ_- is integral. And restricting the prequantization of (Z, Ω) to the brane $\mathcal{L}$ induces a prequantization of F, which forces it to be integral in cohomology.

The other direction is essentially a corollary of Theorem 5.1 from [25], and Proposition 4.1.3 above. First, since σ_- is integral, the periods of Ω_- on the source fibres of the groupoid are integers. This breaks up into two conditions: the periods of $B = \mathrm{Re}(\Omega_-)$ are integer, and those of $\omega = \mathrm{Im}(\Omega_-)$ are zero. By Corollary 5.2 of [25], we can find a prequantization (U_1, ∇_1) of B, and a multiplicative $\mathbb{R}$-line bundle with connection (U_2, ∇_2), such that $(\nabla_2)^2 = 2\pi\omega$. Taking the tensor product, we get a complex multiplicative line bundle (U_-, ∇_-) with curvature given by $-2\pi i\Omega_-$. The curvature is a holomorphic $(2,0)$-form. Hence, the $(0,1)$-component of ∇_- defines a holomorphic structure on U_-, and the $(1,0)$-component defines a holomorphic connection. Hence, we get a holomorphic multiplicative prequantization of Ω_-.

Next, choose a prequantization (L, D) of F on the base B (with complex manifold structures $X_\pm$). We will construct the full prequantization in a similar way to the construction of the Morita equivalence in Proposition 4.1.3. Recall that we first consider the groupoid $(\mathcal{R}, \Omega)$ over the doubled base $X_-^{(2)} \coprod X_-^{(1)}$, coming from the trivial Morita self-equivalence of $(\Sigma(X_-), \Omega_-)$. Since this groupoid consists of 4 copies of $\Sigma(X_-)$, with the same multiplication, it inherits a multiplicative prequantization (U', ∇') from (U_-, ∇_-). Now consider the line bundle (L', D') on the base which is equal to (L, D) on $X_-^{(2)}$, and the trivial bundle $(X_-^{(1)} \times \mathbb{C}, d)$ on $X_-^{(1)}$. Then

$$t^*(L', D') \otimes s^*(L', D')^{-1} \otimes (U', \nabla')$$

is automatically multiplicative, and its curvature exactly coincides with the new holomorphic symplectic form. It is therefore of type $(2, 0)$ with respect to the new complex structure. The connection then induces a holomorphic structure, and a holomorphic connection prequantizing the symplectic form. Hence we get a prequantization of the GK structure. □

Remark 4.4.7. Suppose (U_-, ∇_-) is a holomorphic multiplicative prequantization of $(\Sigma(X_-), \Omega_-)$, and (L, D) is a prequantization of F on the base. Then the prequantization of the Morita equivalence (Z, Ω) is given by

$$(U, \nabla) = t^*(L, D) \otimes (U_-, \nabla_-),$$

and the holomorphic multiplicative prequantization of $(\Sigma(X_+), \Omega_+)$ is given by

$$(U_+, \nabla_+) = t^*(L, D) \otimes s^*(L, D)^{-1} \otimes (U_-, \nabla_-).$$

Example 4.4.8 (Kähler Manifold). Consider a Kähler manifold (M, I, g, ω), and the corresponding Morita equivalence

$$(Z, \Omega) = (T^*X, \Omega_0 + \pi^*\omega).$$

The symplectic groupoid of the zero Poisson structure is given by T^*X, with the canonical symplectic form Ω_0, and this has a multiplicative primitive given by the tautological 1-form θ. Hence, the trivial bundle $T^*X \times \mathbb{C}$, with the connection $d - 2\pi i\theta$ gives a holomorphic multiplicative prequantization. Now let (L, D) be a prequantization of ω. The GK quantization is then given by

$$(\pi^*L, \pi^*(D) - 2\pi i\theta)$$

on the Morita equivalence. Notice that since $\pi : Z \to X$ is holomorphic, $\pi^*\omega$ is a real $(1, 1)$-form, which is cohomologous to the holomorphic $(2, 0)$-form Ω. ■

4.5 Čech description

Recall that in Section 1.2, the construction of the Morita equivalence was given first via a Čech description. It is also possible to do this in the more general case, describing GK structures in terms of a non-linear cohomology theory. In Section 5.2, we give an explicit example of such a construction. In order to do this in general, we preference the holomorphic Poisson structure (X_-, σ_-), and describe the GK structure relative to this.

Definition 4.5.1. Let $(\Sigma(X_-), \Omega_-)$ be the holomorphic symplectic integration of σ_-. We define $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))$ to be the group of brane bisections, and $\mathrm{LBis}(\Sigma(X_-))$ to be the subgroup of holomorphic Lagrangian bisections. We also use the same notation to denote the corresponding presheaves of local bisections, expressed as sections of the source map.

Choose a cover of X_- by open sets W_i, and for each i, choose a brane bisection $\gamma_i \in \mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))(W_i)$. These consist of maps $\gamma_i : W_i \to \Sigma(X_-)$, such that

1. $s \circ \gamma_i = id_{W_i}$,

2. $t \circ \gamma_i = \phi_i$, a diffeomorphism onto its image,

3. the image of γ_i is a brane with respect to Ω_-.

Define $V_i = \phi_i(W_i)$, and let $F_i = \gamma_i^* \Omega_-$, a real-closed 2-form. The map $\phi_i : W_i \to V_i$ only preserves the imaginary part of σ_-, but we can use it to pullback the complex structure. By Theorem 4.2.2, $(I_{+,i} = \phi_i^*(I_-), I_-, Q, F_i)$ defines a degenerate GK structure of symplectic type on W_i.

Let $W_{ij} = W_i \cap W_j$, denote $V_{ij}^{(i)} = \phi_i(W_{ij})$, and define $\delta(\gamma)_{ji} = \gamma_j * \gamma_i^{-1}$. Since, for $y \in W_{ij}$, $s(\gamma_j(y) * \gamma_i(y)^{-1}) = \phi_i(y)$, and $t(\gamma_j(y) * \gamma_i(y)^{-1}) = \phi_j(y)$, it follows that $\delta(\gamma)_{ji} \in \mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))(V_{ij}^{(i)})$, and this bisection induces the diffeomorphism $t \circ \delta(\gamma)_{ji} = \phi_j \circ \phi_i^{-1} : V_{ij}^{(i)} \to V_{ij}^{(j)}$.

Now assume that $\delta(\gamma)_{ji} \in \mathrm{LBis}(\Sigma(X_-))(V_{ij}^{(i)})$, a holomorphic Lagrangian bisection. Then, by Proposition 2.2.11, $\phi_j \circ \phi_i^{-1}$ is holomorphic Poisson, and hence $I_{+,j}|_{W_{ij}} = I_{+,i}|_{W_{ij}}$. Furthermore, since $\gamma_j|_{W_{ij}} = \delta(\gamma)_{ji} * \gamma_i|_{W_{ij}}$, we have that $F_j|_{W_{ij}} = F_i|_{W_{ij}}$. Therefore, the local GK structures glue together to form a global degenerate GK structure of symplectic type (I_+, I_-, Q, F) on X_-.

It is easy to describe the corresponding Morita equivalence with brane bisection in terms of the Čech data. First, using the holomorphic Poisson maps $\phi_j \circ \phi_i^{-1}$, we glue together the open sets V_i into a holomorphic Poisson manifold $X_+ = \coprod V_i / \sim$, which is isomorphic via ϕ_i to X_-, equipped with (I_+, σ_+).

Next, we construct the Morita equivalence (Z, Ω), by gluing together the right-principal $\Sigma(X_-)$-bundles $t^{-1}(V_i) \to V_i$ using the holomorphic Lagrangian bisections $\delta(\gamma)_{ji}$. More precisely, by Proposition 2.2.11, left multiplication by $\delta(\gamma)_{ji}$ defines a holomorphic symplectic isomorphism of right-principal $\Sigma(X_-)$-bundles:

$$L_{ji} : t^{-1}(V_{ij}^{(i)}) \to t^{-1}(V_{ij}^{(j)}), \qquad g \mapsto \delta(\gamma)_{ji}(t(g)) * g.$$

These maps satisfy $s \circ L_{ji} = s$, and $t \circ L_{ji} = \phi_j \circ \phi_i^{-1} \circ t$, as well as the cocycle condition:

$$L_{kj} \circ L_{ji} = L_{ki}.$$

Hence, we can use them to glue $\{t^{-1}(V_i)\}$ into a holomorphic symplectic bundle (Z, Ω) as follows:

$$Z = (\coprod_i t^{-1}(V_i)) / \sim, \quad g_i \sim L_{ji}(g)_j.$$

This has a surjective Poisson submersion $\pi_+ : Z \to X_+$, defined locally by t, and a surjective anti-Poisson submersion $\pi_- : Z \to X_-$, defined locally by s. A priori, this is only a right-principal $\Sigma(X_-)$-bundle. But the local bisections γ_i glue together into a global brane bisection $\gamma : X_- \to Z$, and this induces an isomorphism between (Z, Ω) and the Morita equivalence $(\Sigma(X_-), \Omega_- + t^* F)$.

Note that we actually obtain more than the Morita equivalence with brane bisection (Z, Ω, γ). We also get a cover of X_+ by the open sets V_i, and local holomorphic Lagrangian bisections $\Lambda_i : V_i \to Z$, such that $\pi_+ \circ \Lambda_i = id_{V_i}$, defined by the identity bisections in $t^{-1}(V_i)$. It is this extra data that allows us to extract the Čech description. And by 'modding out' this choice of holomorphic Lagrangian bisections, we obtain a classification of GK structures.

Definition 4.5.2. A *GK Čech cocycle* for (X_-, σ_-) consists of an open cover $X_- = \cup W_i$, and a collection of brane bisections $\gamma_i \in \mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))(W_i)$, such that $\delta(\gamma)_{ji} \in \mathrm{LBis}(\Sigma(X_-))(V_{ij}^{(i)})$. We say that two Čech cocycles γ and η are equivalent if, after passing to a common refinement of their covers, they are related by holomorphic Lagrangian bisections:

$$\eta_i = l_i * \gamma_i,$$

for $l_i \in \mathrm{LBis}(\Sigma(X_-))((t \circ \gamma_i)(W_i))$.

Theorem 4.5.3. *There is a bijection between degenerate GK structures of symplectic type, with fixed holomorphic Poisson manifold (X_-, σ_-), and equivalences classes of GK Čech cocycles for (X_-, σ_-).*

Proof. The preceding discussion shows that there is a GK structure associated to every GK Čech cocycle, and it is straightforward to see this assignment is invariant under refinement of covers. Now suppose that we have two GK Čech cocycles, γ and η, over the same cover, such that $\eta = l * \gamma$, for holomorphic Lagrangian bisections l_i. Then, since $t \circ l_i$ is holomorphic, η and γ induce the same complex structure I_+, and furthermore

$$\eta_i^* \Omega_- = (t \circ \gamma_i)^* l_i^* \Omega_- + \gamma_i^* \Omega_- = \gamma_i^* \Omega_-.$$

Hence, γ and η give rise to the same GK structure, and so there is a well-defined map from equivalence classes of GK Čech cocycles to GK structures on X_-. We will show that this is a bijection.

First, we show injectivity. So let γ and η be two GK Čech cocycles giving rise to the same GK structure. After taking a common refinement, we have $\gamma_i, \eta_i : W_i \to \Sigma(X_-)$, such that $\gamma_i^* \Omega_- = \eta_i^* \Omega_-$. Therefore $\eta_i * \gamma_i^{-1}$ is holomorphic Lagrangian, and so γ and η are equivalent.

Next, we show surjectivity. Let $(Z, \Omega, \mathcal{L})$ be a holomorphic symplectic Morita equivalence with brane bisection between (X_+, σ_+) and (X_-, σ_-). We think of $\mathcal{L}$ as a section of π_-, and let $\phi = \pi_+ \circ \mathcal{L} : X_- \to X_+$. Now choose an open cover of $X_+ = \cup V_i$, such that over each V_i, there is a holomorphic Lagrangian bisection $\Lambda_i : V_i \to Z$, viewed as a section of π_+, and let $W_i = \phi^{-1}(V_i)$. Note that the W_i give an open cover of X_-, since ϕ is a diffeomorphism.

Because the action of $\Sigma(X_-)$ is principal, for each index i, there is a unique map $\gamma_i : W_i \to \Sigma(X_-)$, such that

$$\mathcal{L}(x) * \gamma_i(x)^{-1} = \Lambda_i(\phi(x)),$$

for all $x \in W_i$. One can then check that $s \circ \gamma_i = id_{W_i}$, $t \circ \gamma_i = (\pi_- \circ \Lambda_i) \circ \phi$, and $\gamma_i^* \Omega_- = \mathcal{L}^* \Omega$, so that γ_i define brane bisections $\gamma_i \in \mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))(W_i)$. It also follows that $\delta(\gamma)_{ji}$ are holomorphic Lagrangian, since

$$\delta(\gamma)_{ji}^* \Omega_- = \gamma_j^* \Omega_-|_{W_{ij}} - \gamma_i^* \Omega_-|_{W_{ij}} = 0,$$

or, from the relation $\Lambda_i = \Lambda_j * \delta(\gamma)_{ji}$. Hence, γ_i define a GK Čech cocycle, which gives rise to the same GK structure as $(Z, \Omega, \mathcal{L})$. This finishes the proof of surjectivity. $\square$

We can give a more sheaf-theoretic description of this result. Consider the separated presheaf of sets $\widetilde{GKS}$, whose sections over an open set W are the elements of $\Big(\mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-))/\mathrm{LBis}(\Sigma(X_-))\Big)(W)$, i.e. the right cosets of $\mathrm{LBis}(\Sigma(X_-))$ over W. Thus, for a brane bisection γ over W, the element $[\gamma] \in \widetilde{GKS}(W)$ is set of bisections $\lambda * \gamma$, where λ runs over all holomorphic Lagrangian bisections in $\mathrm{LBis}(\Sigma(X_-))(t \circ \gamma(W))$. We define $\mathcal{GKS}$ to be its sheafification, and $\mathcal{GKS}^+$ to be the subsheaf of sections represented by positive brane bisections.

Corollary 4.5.4. *There is a bijection between $H^0(X_-, \mathcal{GKS}^+)$ (resp. $H^0(X_-, \mathcal{GKS})$), and the set of (resp. degenerate) GK structures of symplectic type, with fixed holomorphic Poisson manifold (X_-, σ_-).*

Example 4.5.5. Let us revisit the Kahler case, where $Q = 0$, and $I_+ = I_- = I$. Then $\mathrm{LBis}(\Sigma(X_-)) = \Omega^{1,cl}$, the sheaf of closed holomorphic 1-forms, and $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X_-)) = \Omega^{1,cl}_{\mathbb{R}}$, the sheaf of real, closed 1-forms. The sheaf $\Omega^{1,cl}$ includes into $\Omega^{1,cl}_{\mathbb{R}}$ as the sheaf of d and d^c-closed forms, by taking the imaginary part. ■

We see that in the Kähler case, we only deal with sheaves of abelian groups, whereas in the general case, the sheaves are highly non-commutative. This gives a precise sense in which GK geometry is a non-linear generalization of Kähler geometry.

Chapter 5

The generalized Kähler potential

In this chapter, we discuss an application of our main result to the problem of locally describing a generalized Kähler structure in terms of a real-valued function analogous to the Kähler potential. There is evidence in the physics literature for the existence of such a *generalized Kähler potential* (see for example [114, 71, 60, 73]). However, all previous constructions only work in a neighbourhood where the Poisson structure Q is regular (i.e. has constant rank). As we will see, by viewing a GK structure as a brane in a holomorphic symplectic Morita equivalence, one can describe the geometry in terms of a real-valued function even in a neighbourhood where the Poisson structure changes rank. The fundamental reason for this is that the brane $\mathcal{L}$ is a Lagrangian submanifold of the real symplectic manifold $(Z, \mathrm{Im}(\Omega))$, and so may be described using a generating function.

Our generalized Kähler potential is obtained in the following way. Let $(Z, \Omega, \mathcal{L})$ be a holomorphic symplectic Morita equivalence with brane bisection. Choose a *Darboux chart* of (Z, Ω), i.e. a local holomorphic symplectomorphism

$$(T^*L, \Omega_0) \to (Z, \Omega),$$

where $L \subset Z$ is a holomorphic Lagrangian submanifold and Ω_0 is the canonical holomorphic symplectic form on the holomorphic cotangent bundle T^*L. Using the identification of T^*L with the bundle of $(1,0)$-forms, the brane $\mathcal{L}$ can be viewed as a smooth submanifold of $T^*_{1,0}L$. We require that the Darboux chart be *generic* in the sense that $\mathcal{L}$ is the graph of a $(1,0)$-form $\eta \in \Omega^{1,0}(L)$. Then, since

$$d\eta = \eta^*\Omega_0 = \Omega|_{\mathcal{L}} = F$$

is a real closed 2-form, we have that $d(\mathrm{Im}(\eta)) = 0$, which locally implies that $\mathrm{Im}(\eta) = -\frac{1}{2}dK$, for $K \in C^\infty(L, \mathbb{R})$ a smooth real-valued function on L, uniquely determined up to a real constant. And since η is a $(1,0)$-form, its real part is determined by its imaginary part, so that $\eta = -i\partial K$ and, finally,

$$F = i\partial\bar{\partial}K.$$

Therefore we call K the *generalized Kähler potential* for the GK structure, and we record these observations in the following theorem.

Theorem 5.0.1. *Let $(Z, \Omega, \mathcal{L})$ be a holomorphic symplectic Morita equivalence with brane bisection. Given a generic Darboux chart $(T^*L, \Omega_0) \to (Z, \Omega)$, there is a smooth real-valued function $K \in C^\infty(L, \mathbb{R})$,* *unique up to an additive real constant, such that* $\mathcal{L} = Gr(-i\partial K)$*. Furthermore, the real* 2*-form* $F = \Omega|_{\mathcal{L}}$ *is given by* $F = i\partial\bar{\partial}K$*. We call* K *the generalized Kähler potential.*

5.1 Examples

In the first example, we see that the generalized Kähler potential does indeed specialize to the classical Kähler potential.

Example 5.1.1 (Kähler potential). Suppose the Poisson structure vanishes: $Q = 0$. Then a GK structure of symplectic type is automatically Kähler, and as we saw in Example 4.3.2, the Morita equivalence is given by a twist of the holomorphic cotangent bundle by the Kähler form ω:

$$(Z, \Omega) = (T^*X, \Omega_0 + \pi^*\omega),$$

and the brane bisection $\mathcal{L}$ is given by the zero section of T^*X viewed as a brane in Z. Now choose a local holomorphic Lagrangian section λ of $\pi : Z \to X$ with domain $U \subset X$ and image L. The projection π defines a holomorphic isomorphism from L to U, and so we can use the natural action of T^*X on Z to define the following holomorphic Darboux chart:

$$(T^*U, \Omega_0) \to (Z, \Omega), \ \alpha_x \mapsto \alpha_x + \lambda(x),$$

where $\alpha_x \in T^*_xU$. Theorem 5.0.1 then tells us that $\mathcal{L} = Gr(-i\partial K)$ and $\omega = i\partial\bar{\partial}K$, for a real-valued function $K \in C^\infty(U, \mathbb{R})$, recovering the usual notion of Kähler potential. ■

In the case where the underlying Poisson structure Q is regular, it is possible to choose the Darboux chart in such a way as to recover the GK potentials occurring in the physics literature:

Example 5.1.2 (Non-degenerate Poisson structure). If the Poisson structure $Q = \omega^{-1}$ is non-degenerate, then both of the holomorphic Poisson structures $\sigma_\pm$ are as well, with corresponding holomorphic symplectic forms

$$\Omega_\pm = \omega I_\pm + i\omega.$$

The Morita equivalence between the resulting holomorphic symplectic manifolds is then

$$(Z, \Omega) = (X_+, \Omega_+) \times (X_-, -\Omega_-),$$

with π_+ and π_- the projections onto the first and second factors, respectively. The brane bisection $\mathcal{L}$ is given by the diagonally embedded manifold. We now construct a holomorphic Darboux chart on Z from a pair of holomorphic Darboux charts for $(X_\pm, \Omega_\pm)$: let $(q^\alpha, p_\alpha)_{\alpha=1}^n$ define a holomorphic Darboux chart on X_+ so that $\Omega_+ = dp_\alpha \wedge dq^\alpha$, and let $(Q^\alpha, P_\alpha)_{\alpha=1}^n$ define a holomorphic Darboux chart on X_- so that $\Omega_- = dP_\alpha \wedge dQ^\alpha$. Then $(q^\alpha, p_\alpha, Q^\alpha, P_\alpha)_{\alpha=1}^n$ defines a holomorphic chart on Z with respect to which the form Ω has the following expression:

$$\Omega = dp_\alpha \wedge dq^\alpha + dQ^\alpha \wedge dP_\alpha.$$

A natural choice for the complex Lagrangian in (Z, Ω) is given by $L = \{Q^\alpha = p_\alpha = 0\}$, so that $(q^\alpha, p_\alpha, Q^\alpha, P_\alpha)_{\alpha=1}^n$ define a holomorphic Darboux chart with q^α and P_α coordinates along L and p_α

and Q^α the fibre coordinates. Theorem 5.0.1 then tells us that in the generic situation, the brane is given in these coordinates by the graph of $-i\partial K$ for a real-valued function $K(q^\alpha, P_\alpha, \bar{q}^\alpha, \bar{P}_\alpha)$:

$$\mathcal{L} = \{(q^\alpha, p_\alpha, Q^\alpha, P_\alpha)_{\alpha=1}^n \mid p_\alpha = -i\frac{\partial K}{\partial q^\alpha},\ Q^\alpha = -i\frac{\partial K}{\partial P_\alpha}\},$$

and the symplectic form is given by $F = i\partial\bar{\partial}K$. Using q^α and P_α as coordinates on this brane, we can then express all of the remaining data in terms of the GK potential K as follows. First, the two projection maps $\pi_\pm$ have the following coordinate expressions

$$\pi_+|_{\mathcal{L}}(q^\alpha, P_\alpha) = (q^\alpha, -i\frac{\partial K}{\partial q^\alpha}), \qquad \pi_-|_{\mathcal{L}}(q^\alpha, P_\alpha) = (-i\frac{\partial K}{\partial P_\alpha}, P_\alpha),$$

and therefore the holomorphic symplectic forms $\Omega_\pm$ can be expressed as follows

$$\begin{aligned}\Omega_+ &= i\Big(\frac{\partial^2 K}{\partial \bar{q}^\beta \partial q^\alpha} dq^\alpha \wedge d\bar{q}^\beta + \frac{\partial^2 K}{\partial P_\beta \partial q^\alpha} dq^\alpha \wedge dP_\beta + \frac{\partial^2 K}{\partial \bar{P}_\beta \partial q^\alpha} dq^\alpha \wedge d\bar{P}_\beta\Big)\\ \Omega_- &= i\Big(\frac{\partial^2 K}{\partial \bar{P}_\beta \partial P_\alpha} d\bar{P}_\beta \wedge dP_\alpha + \frac{\partial^2 K}{\partial q^\beta \partial P_\alpha} dq^\beta \wedge dP_\alpha + \frac{\partial^2 K}{\partial \bar{q}^\beta \partial P_\alpha} d\bar{q}^\beta \wedge dP_\alpha\Big).\end{aligned}$$

Since these forms are holomorphic symplectic, they determine the complex structures $I_\pm$. The symplectic form ω is the common imaginary part of $\Omega_\pm$:

$$\omega = \frac{1}{2}\Big(\frac{\partial^2 K}{\partial P_\beta \partial q^\alpha} dq^\alpha \wedge dP_\beta + \frac{\partial^2 K}{\partial \bar{P}_\beta \partial q^\alpha} dq^\alpha \wedge d\bar{P}_\beta + c.c.\Big).$$

In this way, we recover the expressions found in [114, Section 4.2]. ■

We now use our notion of GK potential to construct new examples of 4-dimensional GK structures for which the Hitchin Poisson structure is *not of constant rank*, dropping from full rank to zero along a codimension 2 submanifold.

Example 5.1.3. Recall from Example 2.2.17 that the Poisson structure $\Pi = x\partial_y \wedge \partial_x$ on $\mathbb{C}^2$ has symplectic groupoid $\mathcal{G} = \mathbb{C}^4$, with source and target maps

$$t(a, b, x, y) = (e^a x, y + xb), \qquad s(a, b, x, y) = (x, y),$$

and symplectic form $\Omega = da \wedge d(y + xb) - db \wedge dx$. Viewing this as the trivial Morita equivalence we can upgrade this to an example of a GK structure by the appropriate choice of a brane bisection. In order to apply Theorem 5.0.1 we choose a Darboux chart for the groupoid as follows:

$$(p_1, p_2, q_1, q_2) = (a, -b, y + xb, x).$$

This puts the symplectic form into the standard form

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

We take q_1, q_2 to be coordinates on the Lagrangian and p_1, p_2 the cotangent fibre coordinates. The source and target maps then have the form

$$t(p_1, p_2, q_1, q_2) = (e^{p_1} q_2, q_1), \qquad s(p_1, p_2, q_1, q_2) = (q_2, q_1 + p_2 q_2).$$

According to Theorem 5.0.1, to define a brane it suffices to choose a real-valued function of q_1 and q_2. We take

$$K(q_1, q_2) = q_1 \bar{q}_1 + q_2 \bar{q}_2.$$

Then the brane is given by the graph of the $(1,0)$-form $-i\partial K = -i(\bar{q}_1 dq_1 + \bar{q}_2 dq_2)$. That is, the brane is given by

$$\mathcal{L} = \{(p_1, p_2, q_1, q_2) \mid p_1 = -i\bar{q}_1, \quad p_2 = -i\bar{q}_2\}.$$

Using q_1 and q_2 as coordinates on $\mathcal{L}$, the source and target maps (restricted to $\mathcal{L}$) are

$$t(q_1, q_2) = (e^{-i\bar{q}_1} q_2, q_1), \qquad s(q_1, q_2) = (q_2, q_1 - i|q_2|^2).$$

Because these are diffeomorphisms, $\mathcal{L}$ is a bisection of s and t. Hence the triple $(\mathcal{G}, \Omega, \mathcal{L})$ defines a degenerate GK structure of symplectic type. By Theorem 5.0.1, the 2-form is given by

$$F = i\partial\bar{\partial}K = i(dq_1 \wedge d\bar{q}_1 + dq_2 \wedge d\bar{q}_2).$$

In these coordinates, the Hitchin Poisson structure is given by

$$Q = 2i(q_2 \partial_{q_1} \wedge \partial_{q_2} - \bar{q}_2 \partial_{\bar{q}_1} \wedge \partial_{\bar{q}_2}),$$

and the complex structures $I_\pm$ can be specified by their holomorphic coordinate functions: the I_+-holomorphic functions are given by $t^*x = e^{-i\bar{q}_1} q_2$ and $t^*y = q_1$, while the I_--holomorphic functions are given by $s^*x = q_2$ and $s^*y = q_1 - i|q_2|^2$.

In order to see that this defines a GK structure, we must determine whether, and where, the induced metric g is positive definite. First we determine where g is invertible: by Lemma 3.1.4 this happens precisely where $\omega_- = gI_- = F^{(1,1)_-}$ is invertible, where $F^{(1,1)_-}$ is the $(1,1)$-component of F with respect to the I_- complex structure. We have

$$F^{(1,1)_-} = i dq_1 \wedge d\bar{q}_1 + i(1 - 2|q_2|^2) dq_2 \wedge d\bar{q}_2 + \bar{q}_2 dq_2 \wedge dq_1 + q_2 d\bar{q}_2 \wedge d\bar{q}_1,$$

so that

$$F^{(1,1)_-} \wedge F^{(1,1)_-} = -2(1 - |q_2|^2) dq_1 \wedge d\bar{q}_1 \wedge dq_2 \wedge d\bar{q}_2.$$

Hence g is invertible whenever $|q_2| \neq 1$. Along $q_2 = 0$, we have $s^*(dx \wedge dy) = dq_2 \wedge dq_1$, so that I_- coincides with the standard complex structure in coordinates (q_1, q_2), and $F^{(1,1)_-} = i dq_1 \wedge d\bar{q}_1 + i dq_2 \wedge d\bar{q}_2$. Since these formulas coincide with the standard Kähler structure on $\mathbb{C}^2$, we know that g is positive definite when $q_2 = 0$ and therefore in the region $|q_2| < 1$. Therefore this defines a generalized Kähler structure on $\mathbb{C} \times D$, where $D = \{q_2 \mid |q_2| < 1\}$. The metric has the explicit form

$$g = 2(dq_1 d\bar{q}_1 + dq_2 d\bar{q}_2 + i\bar{q}_2 dq_1 dq_2 - iq_2 d\bar{q}_1 d\bar{q}_2).$$

Note that if we restrict to the disc $\{q_1 = c\} \times D$ the metric pulls back to $2dq_2 d\bar{q}_2$, and so this disc has finite volume. Therefore the metric is not complete by the Hopf-Rinow theorem. ■

Example 5.1.4. We now generalize the previous example by choosing a different generalized Kähler potential K, of the following form:

$$K(q_1, q_2) = a(q_1, \bar{q}_1) + b(q_2, \bar{q}_2),$$

for real-valued functions a and b. The brane is then given by

$$\mathcal{L} = \{(p_1, p_2, q_1, q_2) \mid p_1 = -i\partial_{q_1} a, \quad p_2 = -i\partial_{q_2} b\}.$$

Again, t and s restrict to $\mathcal{L}$ to be diffeomorphisms, and hence $\mathcal{L}$ defines a brane bisection, defining a degenerate GK structure. In terms of the real-valued functions

$$\alpha(q_1, \bar{q}_1) = \frac{\partial^2 a}{\partial q_1 \partial \bar{q}_1}, \qquad \beta(q_2, \bar{q}_2) = \frac{\partial^2 b}{\partial q_2 \partial \bar{q}_2},$$

we have

$$F = i(\alpha dq_1 \wedge d\bar{q}_1 + \beta dq_2 \wedge d\bar{q}_2),$$

which has $(1,1)$ component with respect to I_- given by

$$F^{(1,1)_-} = i\alpha dq_1 \wedge d\bar{q}_1 + i\beta(1 - 2\alpha\beta|q_2|^2) dq_2 \wedge d\bar{q}_2 + \alpha\beta\bar{q}_2 dq_2 \wedge dq_1 + \alpha\beta q_2 d\bar{q}_2 \wedge d\bar{q}_1,$$

so that

$$F^{(1,1)_-} \wedge F^{(1,1)_-} = -2\alpha\beta(1 - \alpha\beta|q_2|^2) dq_1 \wedge d\bar{q}_1 \wedge dq_2 \wedge d\bar{q}_2.$$

Furthermore, the induced metric is given by

$$g = 2(\alpha dq_1 d\bar{q}_1 + \beta dq_2 d\bar{q}_2 + i\alpha\beta\bar{q}_2 dq_1 dq_2 - i\alpha\beta q_2 d\bar{q}_1 d\bar{q}_2).$$

We want the metric to be positive definite. Setting $q_2 = 0$ shows that we must have $\alpha, \beta > 0$. The expression for $F^{(1,1)_-} \wedge F^{(1,1)_-}$ shows that g will be invertible precisely when

$$1 \neq \alpha\beta|q_2|^2.$$

Setting $q_2 = 0$ shows that we must therefore require that $\alpha\beta|q_2|^2 < 1$. Finally, because α and β depend only on $q_1, \bar{q}_1$ and $q_2, \bar{q}_2$ respectively, α must be bounded by a constant and β must be bounded by $\frac{C}{|q_2|^2}$, for C a positive constant. Under these assumptions, we obtain examples of generalized Kähler structures of symplectic type. ■

Proposition 5.1.5. *In the context of the previous example, choose the generalized Kähler potential to be*

$$K(q_1, q_2) = \frac{|q_1|^2}{C} - Li_2(-|q_2|^2),$$

for $C > 1$ a constant, where $Li_2(z) = -\int_0^z log(1-u)\frac{du}{u}$ is the dilogarithm. Then we get $\alpha = \frac{1}{C}$ and

$\beta = \frac{1}{1+|q_2|^2}$, *and hence the metric is given by*

$$g = 2(\frac{1}{C}dq_1 d\bar{q}_1 + \frac{1}{1+|q_2|^2}(dq_2 d\bar{q}_2 + \frac{i}{C}\bar{q}_2 dq_1 dq_2 - \frac{i}{C}q_2 d\bar{q}_1 d\bar{q}_2)).$$

This gives a generalized Kähler structure on $\mathbb{R}^4$ for which the metric is complete.

Proof. After the argument of Example 5.1.4, we need only show the completeness of the metric. Note that the metric is translation invariant in the q_1 direction. Quotienting by a $\mathbb{Z}^2$ lattice, we obtain a metric on $S^1 \times S^1 \times \mathbb{C}$. Completeness of this metric is equivalent to completeness of the original. To show completeness, we need only investigate geodesics escaping to infinity in the q_2-direction. In coordinates $q_1 = x + iy$ and $q_2 = \sinh(s)e^{i\theta}$, the metric has the following form:

$$g = \frac{2}{C}(dx^2 + dy^2) + 2(ds^2 + \tanh^2(s)d\theta^2 - \frac{2}{C}\tanh^2(s)dxd\theta - \frac{2}{C}\tanh(s)dyds).$$

For any geodesic $\gamma(t) = (x(t), y(t), s(t), \theta(t))$, we have

$$\sqrt{g(\dot{\gamma}, \dot{\gamma})} \geq \sqrt{\frac{2(C-1)}{C}}\dot{s},$$

from which it follows that the length of the curve over the interval $[t_0, t_1]$ is bounded below in the following way:

$$L(\gamma) \geq \sqrt{\frac{2(C-1)}{C}}(s(t_1) - s(t_0)).$$

Therefore the length of a curve that escapes to infinity is unbounded and so the metric is complete. □

Example 5.1.6. We now construct GK metrics for the Poisson structure $\sigma = xy\partial_y \wedge \partial_x$ on $\mathbb{C}^2$. Recall from Example 2.2.18 that this Poisson structure has symplectic groupoid $\mathcal{G} = \mathbb{C}^4$, with source and target maps

$$t(a, b, x, y) = (x\exp(yb), y\exp(-xa)), \qquad s(a, b, x, y) = (x, y),$$

and symplectic form $\Omega = da \wedge dx + db \wedge dy + d(xa) \wedge d(yb)$. We again view this as the trivial Morita equivalence. A Darboux chart is given by

$$(p_1, p_2, q_1, q_2) = (a\exp(-yb), b, x\exp(yb), y),$$

so that the form is given in these coordinates by

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

The source and target maps are then given by

$$t(p_1, p_2, q_1, q_2) = (q_1, q_2\exp(-q_1 p_1)), \qquad s(p_1, p_2, q_1, q_2) = (q_1\exp(-q_2 p_2), q_2).$$

We now add a slight twist: we scale the Poisson structure by a real number t: $t\sigma$. The only thing that changes in our description of $\mathcal{G}$ is the source, target and multiplication maps. The deformed source and target maps are given by

$$t(p_1, p_2, q_1, q_2) = (q_1, q_2 \exp(-tq_1 p_1)), \qquad s(p_1, p_2, q_1, q_2) = (q_1 \exp(-tq_2 p_2), q_2).$$

Notice that when $t = 0$, the source and target maps are just projection onto the q-coordinates, which is what we expect for the zero Poisson structure.

We now construct a GK metric by choosing a potential function according to Theorem 5.0.1. As before, q_1, q_2 are coordinates along the holomorphic Lagrangian, and p_1, p_2 are the cotangent fibre coordinates. Therefore, we must choose a real-valued function of q_1 and q_2. Let us choose a potential function which only depends on the radial components:

$$K(q_1, q_2) = f(|q_1|^2, |q_2|^2),$$

for a real-valued function f. The brane is given by

$$\mathcal{L} = \{(p_1, p_2, q_1, q_2) \mid p_1 = -i\bar{q}_1 f_1, \quad p_2 = -i\bar{q}_2 f_2\},$$

where f_i denotes the derivative of f with respect to the i^{th}-coordinate. Using q_1, q_2 as coordinates on $\mathcal{L}$, the restricted source and target maps are then

$$t(q_1, q_2) = (q_1, q_2 \exp(it|q_1|^2 f_1)), \qquad s(q_1, q_2) = (q_1 \exp(it|q_2|^2 f_2), q_2).$$

Notice that the arguments of the exponents only depend on the radial components. Furthermore, the arguments are purely imaginary, so that s and t do not distort the radial components. Hence, it is easy to invert these maps, and so they are diffeomorphisms. It then follows by Theorem 5.0.1 that $(\mathcal{G}, \Omega, \mathcal{L})$ defines a degenerate GK structure of symplectic type. The 2-form F is given by

$$F = iA dq_1 \wedge d\bar{q}_1 + iB dq_2 \wedge d\bar{q}_2 + iC dq_1 \wedge d\bar{q}_2 + i\bar{C} dq_2 \wedge d\bar{q}_1,$$

where

$$A = \frac{\partial^2 K}{\partial q_1 \partial \bar{q}_1} = f_1 + |q_1|^2 f_{11}, \qquad B = \frac{\partial^2 K}{\partial q_2 \partial \bar{q}_2} = f_2 + |q_2|^2 f_{22}, \qquad C = \frac{\partial^2 K}{\partial q_1 \partial \bar{q}_2} = \bar{q}_1 q_2 f_{21}.$$

This structure has a high-degree of symmetry. Indeed, the group $S^1 \times S^1$ acts on $\mathbb{C}^2$ by rotating each coordinate. It is easy to see that $t\sigma$ and K and F are invariant, and that the source and target maps are equivariant. Therefore we get an $S^1 \times S^1$-invariant degenerate GK structure of symplectic type.

Let us now investigate the positivity of the metric. Appealing again to Lemma 3.1.4, which allows us to express $\omega_- = gI_- = F^{(1,1)_-}$, we can determine whether the metric is non-degenerate by studying the invertibility of the $(1,1)$-component of F with respect to I_-. We have

$$\begin{aligned}\omega_- =& iA dq_1 \wedge d\bar{q}_1 + iB(1 - 2t^2|q_1|^2|q_2|^2 D) dq_2 \wedge d\bar{q}_2 \\ &+ iC(1 - t^2|q_1|^2|q_2|^2 D) dq_1 \wedge d\bar{q}_2 - i\bar{C}(1 - t^2|q_1|^2|q_2|^2 D) d\bar{q}_1 \wedge dq_2 \\ &+ t\bar{q}_1\bar{q}_2 D(1 + it\bar{q}_1 q_2 \bar{C}) dq_1 \wedge dq_2 + tq_1 q_2 D(1 - itq_1\bar{q}_2 C) d\bar{q}_1 \wedge d\bar{q}_2,\end{aligned}$$

where $D = AB - |C|^2$, so that

$$Vol_{g_t} = \frac{\omega_-^2}{2} = D(1 - t^2|q_1|^2|q_2|^2 D)(idq_1 \wedge d\bar{q}_1) \wedge (idq_2 \wedge d\bar{q}_2).$$

First set $t = 0$, so that the Poisson structure vanishes. Then $I_+ = I_- = I_0$, the complex structure of the holomorphic Lagrangian, $F = \omega_- = \omega_+$, and

$$Vol_{g_0} = D(idq_1 \wedge d\bar{q}_1) \wedge (idq_2 \wedge d\bar{q}_2),$$

so that D is the determinant of the complex Hessian of K, with respect to the complex structure I_0. We therefore see that metric is non-degenerate if and only if $D \neq 0$, and is positive definite if and only if K has positive definite Hessian. In this case, we get a Kähler structure on $\mathbb{C}^2$.

For general t, the condition for non-degeneracy is that $D(1 - t^2|q_1|^2|q_2|^2 D) \neq 0$, which breaks up into the two conditions

$$D \neq 0, \qquad D < \frac{1}{t^2|q_1|^2|q_2|^2},$$

which must hold for all values of q_1, q_2. Note that if the second condition holds for some value of t, then it also holds for all values with smaller norm, and hence for the path from 0 to t. Since the signature of a non-degenerate family of symmetric tensors is constant, this means that g_t has the same signature as g_0.

Lemma 5.1.7. *The metric g_t of the degenerate GK structure is positive definite if and only if K has positive definite Hessian with respect to I_0 and*

$$D < \frac{1}{t^2|q_1|^2|q_2|^2}, \tag{5.1}$$

for all q_1, q_2.

If the condition of the lemma is satisfied for $t = 1$, then we get a family of GK structures, interpolating between a Kähler structure at $t = 0$, and a GK structure with non-trivial Hitchin Poisson bracket $\sigma = xy\partial_y \wedge \partial_x$ at $t = 1$. Note that condition 5.1 will eventually be violated for large enough finite t: the family only exists for a finite range of t. Finally, it is possible to write an explicit expression for the metric, in terms of K:

$$\begin{aligned}\frac{1}{2}g_t = &(1 + 2t^2|q_1|^2|q_2|^2|C|^2)(Adq_1 d\bar{q}_1 + Bdq_2 d\bar{q}_2)\\ &+(1 + t^2|q_1|^2|q_2|^2(AB + |C|^2))(Cdq_1 d\bar{q}_2 + \bar{C}d\bar{q}_1 dq_2)\\ &-it\bar{q}_1\bar{q}_2(1 + it\bar{q}_1 q_2\bar{C})(ACdq_1^2 + B\bar{C}dq_2^2 + (AB + |C|^2)dq_1 dq_2)\\ &+itq_1q_2(1 - itq_1\bar{q}_2 C)(A\bar{C}d\bar{q}_1^2 + BCd\bar{q}_2^2 + (AB + |C|^2)d\bar{q}_1 d\bar{q}_2).\end{aligned}$$

■

Example 5.1.8. Let us consider a special case of the previous example, where we choose the potential to be

$$K(q_1, q_2) = -Li_2(-|q_1|^2) - Li_2(-|q_2|^2),$$

for $Li_2(z) = -\int_0^z \log(1-u)\frac{du}{u}$, the dilogarithm function. This example will proceed in a similar way to

Proposition 5.1.5. We find that

$$A = \frac{1}{1+|q_1|^2}, \qquad B = \frac{1}{1+|q_2|^2}, \qquad C = 0,$$

so that

$$F = i\frac{dq_1 \wedge d\bar{q}_1}{1+|q_1|^2} + i\frac{dq_2 \wedge d\bar{q}_2}{1+|q_2|^2},$$

and

$$\frac{1}{2}g_t = \frac{dq_1 d\bar{q}_1}{1+|q_1|^2} + \frac{dq_2 d\bar{q}_2}{1+|q_2|^2} + it\frac{q_1 q_2 d\bar{q}_1 d\bar{q}_2 - \bar{q}_1\bar{q}_2 dq_1 dq_2}{(1+|q_1|^2)(1+|q_2|^2)}.$$

We see that g_0 is positive definite, and condition 5.1 holds if and only if $t^2 \leq 1$. Hence we have a family of GK metrics for $0 \leq t \leq 1$. Choosing coordinates $q_1 = \sinh(s_1)e^{i\theta_1}$ and $q_2 = \sinh(s_2)e^{i\theta_2}$, the metric has the form

$$\frac{1}{2}g_t = ds_1^2 + \tanh^2(s_1)d\theta_1^2 + ds_2^2 + \tanh^2(s_2)d\theta_2^2 + 2t\tanh(s_1)\tanh(s_2)(\tanh(s_1)d\theta_1 ds_2 + \tanh(s_2)ds_1 d\theta_2).$$

For small values of s_1, s_2, the metric looks approximately Euclidean, whereas for large s_1 and s_2, it is approximately given by

$$\frac{1}{2}g_t \approx (ds_1^2 + d\theta_2^2 + 2t ds_1 d\theta_2) + (ds_2^2 + d\theta_1^2 + 2t d\theta_1 ds_2),$$

which is a product of two flat cylinders.

Proposition 5.1.9. *The metric g_t is complete for $0 \leq t < 1$, and non-complete for $t = 1$.*

Proof. First, for $t = 1$, we will show that the metric is non-complete by giving an example of a finite-length path escaping to infinity. Consider the path $\gamma(\tau)$ given by

$$(s_1(\tau), s_2(\tau), \theta_1(\tau), \theta_2(\tau)) = (\tau, \tau, -\log(\cosh(\tau)), -\log(\cosh(\tau))),$$

for $1 \leq \tau \leq \infty$. Then $g_1(\dot{\gamma}, \dot{\gamma}) = 4(1 - \tanh^4(\tau))$, so that the length is

$$L(\gamma) = 2\int_1^\infty \sqrt{1 - \tanh^4(\tau)}d\tau < \infty.$$

Next, for $t < 1$, we show that the metric is complete by giving a lower bound on the length of any path. Given a path

$$\gamma(\tau) = (s_1(\tau), s_2(\tau), \theta_1(\tau), \theta_2(\tau)),$$

we have

$$\begin{aligned}\sqrt{g_t(\dot{\gamma}, \dot{\gamma})} &\geq \sqrt{2(1 - t^2\tanh^2(s_1)\tanh^2(s_2))(\dot{s_1}^2 + \dot{s_2}^2)} \\ &\geq \sqrt{2}(1-t^2)^{\frac{1}{2}}\sqrt{\dot{s_1}^2 + \dot{s_2}^2},\end{aligned}$$

from which it follows that the length of the curve over the interval $[a, b]$ is bounded below in the following way:

$$L(\gamma) \geq \sqrt{2}(1-t^2)^{\frac{1}{2}} \int_a^b \sqrt{\dot{s_1}^2 + \dot{s_2}^2} d\tau \geq \sqrt{2}(1-t^2)^{\frac{1}{2}} \sqrt{(s_1(b) - s_1(a))^2 + (s_2(b)^2 - s_2(a)^2)}.$$

Therefore, the length of any path escaping to infinity is unbounded. □

■

5.2 Toric GK surfaces

We can apply the calculations of Example 5.1.6 to give constructions, in the spirit of Section 4.5, of generalized Kähler deformations of toric Kahler surfaces. See [51] for an introduction to toric geometry. Let (X, I, ω, g) be a toric Kahler surface: X is a compact connected complex 2-dimensional Kähler manifold, with an effective action of $\mathbb{T}^2 = S^1 \times S^1$, which is Hamiltonian for ω, and which preserves I. This action then complexifies to give an effective holomorphic action of $\mathbb{C}^* \times \mathbb{C}^*$. X can be covered by holomorphic charts, which are isomorphic to $\mathbb{C}^2$, with the action of $(\mathbb{C}^*)^2$ given by

$$(\mu, \nu) * (x, y) = (\mu^{n_1} \nu^{m_1} x, \mu^{n_2} \nu^{m_2} y),$$

where (n_1, n_2) and (m_1, m_2) give a basis of $\mathbb{Z}^2$. This means that $n_1 m_2 - n_2 m_1 = \pm 1$, and by flipping the coordinates, we can assume that this value is 1. We can furthermore assume that one of the charts U_0 is given by $\mathbb{C}^2 = \{(q_1, q_2)\}$, with the standard action of $(\mathbb{C}^*)^2$. The $(\mathbb{C}^*)^2$ action is generated by 2 holomorphic vector fields, V_1, V_2, given in the U_0 chart by

$$V_1 = q_1 \partial_{q_1}, \qquad V_2 = q_2 \partial_{q_2}.$$

The bivector $\sigma = V_2 \wedge V_1$ defines a holomorphic Poisson structure. In each of the charts, it has the form $\sigma = xy\partial_y \wedge \partial_x$, and we can see that it vanishes precisely where the action fails to be free.

Symplectic groupoid

We now construct the symplectic groupoid integrating $t\sigma$. The orbits of the groupoid are given by the symplectic leaves of the Poisson structure. For $t \neq 0$, there is a single 2-dimensional open symplectic leaf, isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, where the action is free, and infinitely many 0-dimensional symplectic leaves where σ vanishes. We can see that each of the above charts are unions of orbits. Lets first restrict to U_0, where $\sigma = tq_1 q_2 \partial_{q_2} \wedge \partial_{q_1}$. We are now in the setting of Examples 2.2.18 and 5.1.6. The groupoid is given by $\mathcal{G}_0 = \mathbb{C}^4 = T^*U_0$, with its canonical symplectic form Ω_0, and with source and target maps given by

$$t(p_1, p_2, q_1, q_2) = (q_1, q_2 \exp(-tq_1 p_1)), \qquad s(p_1, p_2, q_1, q_2) = (q_1 \exp(-tq_2 p_2), q_2).$$

We can write these in a more invariant way: the vector fields V_1 and V_2 define holomorphic functions $\tilde{V}_1$ and $\tilde{V}_2$ on T^*X. In terms of these functions, we see that

$$t(\alpha) = (1, \exp(-t\tilde{V}_1(\alpha))) * \pi(\alpha), \qquad s(\alpha) = (\exp(-t\tilde{V}_2(\alpha)), 1) * \pi(\alpha),$$

where α denotes an element of the cotangent bundle, π is the bundle projection, and $*$ denotes the action of $(\mathbb{C}^*)^2$ on X. Furthermore, the multiplication is given by

$$m(\beta,\alpha) = (1, \exp(t\tilde{V_1}(\alpha))) * \beta + (\exp(t\tilde{V_2}(\beta)), 1) * \alpha,$$

the identity bisection is given by the zero section, and the inverse of an element α is given by

$$\iota(\alpha) = (\exp(-t\tilde{V_2}(\alpha), \exp(-t\tilde{V_1}(\alpha)))) * (-\alpha).$$

In these expressions, $*$ denotes the action of $(\mathbb{C}^*)^2$ on T^*X. But now note that all the above formulas are well-defined on all of T^*X. Hence, this defines the symplectic integration of $t\sigma$.

If we switch into another chart V, where the action is given by weights $(n_1, n_2), (m_1, m_2)$, then

$$V_1 = n_1 x\partial_x + n_2 y\partial_y, \qquad V_2 = m_1 x\partial_x + m_2 y\partial_y,$$

and so the source and target maps now have the form

$$\begin{aligned} t(p_x, p_y, x, y) &= (\exp(-tm_1(n_1 xp_x + n_2 yp_y))x, \exp(-tm_2(n_1 xp_x + n_2 yp_y))y), \\ s(p_x, p_y, x, y) &= (\exp(-tn_1(m_1 xp_x + m_2 yp_y)x, \exp(-tn_2(m_1 xp_x + m_2 yp_y))y). \end{aligned}$$

Generalized Kähler potentials

The symplectic groupoid T^*X comes equipped with extra data: i.e. it is a cotangent bundle. Therefore, we can use potential functions to define brane bisections.

Lemma 5.2.1. *Let V be a toric chart, isomorphic to $\mathbb{C}^2$, with $(\mathbb{C}^*)^2$ action given by weights (n_1, n_2) and (m_1, m_2), and let T^*V be the restriction of the symplectic groupoid to this chart. Let K be a real-valued $\mathbb{T}^2$-invariant function. Then $\mathcal{L} = Gr(-i\partial K)$ defines a brane bisection.*

Proof. The potential function has the form

$$K(x, y) = f(|x|^2, |y|^2),$$

for a real-valued function f. The submanifold $\mathcal{L} = Gr(-i\partial K)$ is given by

$$p_x = -i\bar{x} f_1(|x|^2, |y|^2), \qquad p_y = -i\bar{y} f_2(|x|^2, |y|^2),$$

where f_i denotes the derivative with respect to the i^{th}-coordinate. $\mathcal{L}$ is automatically a brane, and therefore we only need to check that it is a bisection, which means that the source and target maps restrict to diffeomorphisms. Using the coordinates (x, y) as coordinates on $\mathcal{L}$, we see that the restricted source and target maps are given by

$$\begin{aligned} t(x, y) &= (\exp(itm_1(n_1|x|^2 f_1 + n_2|y|^2 f_2))x, \exp(itm_2(n_1|x|^2 f_1 + n_2|y|^2 f_2))y), \\ s(x, y) &= (\exp(itn_1(m_1|x|^2 f_1 + m_2|y|^2 f_2)x, \exp(itn_2(m_1|x|^2 f_1 + m_2|y|^2 f_2))y). \end{aligned}$$

The key point here is that the arguments of the exponential functions are purely imaginary, and only depend on the radial components. Therefore, they can be inverted. ☐

Lemma 5.2.2. *Consider a toric Kähler structure ω on $\mathbb{C}^2$. There exists a $\mathbb{T}^2$-invariant Kähler potential K, which is unique up to a real additive constant.*

Proof. The action is given by a map

$$\rho : \mathbb{T}^2 \to \mathrm{Symp}(\mathbb{C}^2, \omega).$$

If K is a Kähler potential, we can average it over $\mathbb{T}^2$ to get

$$\tilde{K} = \int_{\mathbb{T}^2} \rho_\theta^*(K) d\theta,$$

which is manifestly $\mathbb{T}^2$-invariant. Then since ρ_θ preserves the complex structure,

$$i\partial\bar{\partial}\tilde{K} = \int_{\mathbb{T}^2} \rho_\theta^*(i\partial\bar{\partial}K) d\theta = \int_{\mathbb{T}^2} \rho_\theta^*(\omega) d\theta = \omega.$$

Now suppose that K_1 and K_2 are two $\mathbb{T}^2$-invariant Kähler potentials. Then

$$K_1 - K_2 = \phi + \bar{\phi},$$

for a holomorphic function ϕ. Hence, $K_1 - K_2$ is a $\mathbb{T}^2$-invariant harmonic function, and hence constant by the maximum principle. □

Theorem 5.2.3. *Let (X, I, ω, g) be a toric Kähler surface, let V_1 and V_2 be the generators of the $(\mathbb{C}^*)^2$-action, and let $\sigma = V_2 \wedge V_1$ be a holomorphic Poisson structure. There is a family of $\mathbb{T}^2$-invariant degenerate GK structures $(X, \omega, I_{+,t}, I_{-,t}, Q_t)$, for $t \in \mathbb{R}$, deforming the Kähler structure, such that*

$$\frac{d}{dt}|_{t=0}(I_{-,t}) = i_{V_2}\omega \otimes V_1 \in H^1(T_X), \qquad \frac{d}{dt}|_{t=0}(I_{+,t}) = i_{V_1}\omega \otimes V_2 \in H^1(T_X).$$

The metric is invertible if and only if

$$t^2 < \frac{2}{\max_X(\langle \omega^2, \sigma \wedge \bar{\sigma} \rangle)},$$

in which case it is positive-definite, defining a GK structure of symplectic type.

Proof. Let U_i be a cover by toric open sets isomorphic to $\mathbb{C}^2$. Over each U_i, we choose a $\mathbb{T}^2$-invariant Kähler potential and consider the brane bisection $\mathcal{L}_i = Gr(-i\partial K_i)$. Note that this brane is uniquely determined by ω, since K_i is unique up to a constant. By Theorem 4.2.2, each brane $\mathcal{L}_i$ inherits a degenerate GK structure of symplectic type. Using the bundle projection, these induce GK structures on each of the open sets U_i. We claim that these glue together to define a global degenerate GK structure on X. First of all, the 2-form is just given by

$$F_i = i\partial\bar{\partial}K_i = \omega,$$

which is globally defined. Next, we show that the complex structures glue together. Lets move into the chart $U_0 = \mathbb{C}^2$. The potential functions K_i are given in this chart by

$$K_i(q_1, q_2) = f^{(i)}(|q_1|^2, |q_2|^2),$$

and these are defined on an open set containing $(\mathbb{C}^*)^2$. The source and target maps associated to $\mathcal{L}_i$ are given by

$$t_i(q_1, q_2) = (q_1, \exp(it|q_1|^2 f_1^{(i)}(|q_1|^2, |q_2|^2))q_2),$$
$$s_i(q_1, q_2) = (\exp(it|q_2|^2 f_2^{(i)}(|q_1|^2, |q_2|^2))q_1, q_2).$$

We need to show that $s_i^*(I) = s_j^*(I)$, or equivalently, that $s_j \circ s_i^{-1}$ is holomorphic, and similarly for the target maps. One checks that

$$\begin{aligned} s_j \circ s_i^{-1}(q_1, q_2) &= (\exp(it|q_2|^2(f_2^{(j)} - f_2^{(i)})(|q_1|^2, |q_2|^2))q_1, q_2) \\ &= (\exp(itq_2\partial_{q_2}(K_j - K_i)(q_1, q_2))q_1, q_2) \\ &= (\exp(itV_2(K_j - K_i))q_1, q_2). \end{aligned}$$

But $V_2(K_j - K_i)$ is holomorphic, since $\bar{\partial}\partial(K_j - K_i) = 0$. Therefore $s_j \circ s_i^{-1}$ is holomorphic. In fact, $V_2(K_j - K_i)$ is a holomorphic real-valued function on $(\mathbb{C}^*)^2$, and therefore it is a constant. Similarly,

$$t_j \circ t_i^{-1}(q_1, q_2) = (q_1, \exp(itV_1(K_j - K_i))q_2),$$

where $V_1(K_j - K_i)$ is a real-valued constant. This shows that the complex structures glue together, and hence we get a family of degenerate GK structures $(\omega, I_{+,t}, I_{-,t})$ on X. Since the maps t_i and s_i are $\mathbb{T}^2$-equivariant, it follows that this GK structure is $\mathbb{T}^2$-invariant.

The deformation $I_{-,t}$ is given infinitesimally by the Čech cocycle $\langle i\partial(K_j - K_i), V_2\rangle V_1$, which is equivalent to the class $i_{V_2}\omega \otimes V_1 \in H^1(T_X)$. Similarly, the deformation $I_{+,t}$ is given infinitesimally by $i_{V_1}\omega \otimes V_2 \in H^1(T_X)$.

Finally, we need to determine for which values of t the metric is positive-definite. Recall from Example 5.1.6 the expression for the volume form of the metric in the chart U_0:

$$Vol_{g_t} = \frac{\omega_-^2}{2} = D(1 - t^2|q_1|^2|q_2|^2 D)(idq_1 \wedge d\bar{q}_1) \wedge (idq_2 \wedge d\bar{q}_2).$$

This can be written invariantly as

$$Vol_{g_t} = (1 - t^2\langle Vol_{g_0}, \sigma \wedge \bar{\sigma}\rangle)Vol_{g_0} = \frac{1}{2}(1 - \frac{t^2}{2}\langle \omega^2, \sigma \wedge \bar{\sigma}\rangle)\omega^2.$$

In order for g_t to remain invertible, this quantity must be non-zero. Since σ vanishes where the action is not free, we therefore have the condition that

$$\frac{t^2}{2}\langle \omega^2, \sigma \wedge \bar{\sigma}\rangle < 1,$$

over all X. Hence, g_t is invertible if and only if

$$t^2 < \frac{2}{\max_X(\langle \omega^2, \sigma \wedge \bar{\sigma}\rangle)}.$$

In this case, g_t and g_0 have the same signature, which implies that g_t is positive-definite. □

Remark 5.2.4. This construction is slightly different from the construction outlined in Section 4.5. In particular, both Poisson structures get deformed in this construction. The advantage is that the 2-form ω does not vary. In fact, it is also possible to cary out the Čech construction of Section 4.5 for a toric Kähler surface (X, I, ω, g) since, as one can check, for a pair of invariant Lagrangian branes $\mathcal{L}_i, \mathcal{L}_j$, one has $\mathcal{L}_i * (\mathcal{L}_j)^{-1} = \mathcal{L}_i - \mathcal{L}_j$. In this case, I_- remains fixed, and I_+ changes in the direction

$$\sigma(\omega) = i_{V_2}\omega \otimes V_1 - i_{V_1}\omega \otimes V_2,$$

which gives the deformation of the holomorphic Poisson structure described by Hitchin [58]. Hence we see that the deformations of Theorem 5.2.3 split the Hitchin deformation into two components.

Remark 5.2.5. Theorem 5.2.3 is similar to a result of Boulanger [10, Corollary 1], which gives a method, based on *generalized symplectic potentials*, for deforming a toric Kähler structure into a toric GK structure using a 'small enough' Poisson structure. Such results are realizations of the general unobstructedness result of Goto [39]. The toric GK structures considered by Boulanger are *anti-diagonal*, but the GK structures of Theorem 5.2.3 are not. One important difference is that the Hitchin Poisson structures of Theorem 5.2.3 are of the form $t\sigma$ for t real, whereas for anti-diagonal toric GK structures, t must be imaginary. It seems that generalized symplectic potentials are optimized for anti-diagonal GK structures where t is imaginary, while generalized Kähler potentials are optimized for the case where t is real. Recently, Wang [105, 106] has extended the work of Boulanger, giving a description of general toric GK structures in terms of a symplectic potential, and a pair of anti-symmetric constant matrices. The special case of symmetric toric GK structures seems to recover the examples of this section. In particular, Section 7 of [106] seems closely related to Example 5.2.7 below.

Corollary 5.2.6. *A toric GK surface (I_+, I_-, Q, ω) is prequantizable if and only if ω is integral in cohomology.*

Proof. By Theorem 4.4.5 we only need to check that the holomorphic Poisson structures are integral. But the symplectic leaves consist of 0-dimensional points and an open leaf isomorphic to $(\mathbb{C}^*)^2$, and both have trivial second homotopy group. □

Example 5.2.7. Consider $\mathbb{P}^1 \times \mathbb{P}^1$ with Kähler form $\omega = n\omega_{FS,1} + m\omega_{FS,2}$, where $\omega_{FS,i}$ is the Fubini-Study form on the i^{th} copy of $\mathbb{P}^1$. In a chart $\mathbb{C}^2$, with standard $\mathbb{T}^2$ action, the Kähler potential is given by

$$K(q_1, q_2) = \frac{n}{2\pi}\log(1 + |q_1|^2) + \frac{m}{2\pi}\log(1 + |q_2|^2).$$

This defines a GK deformation by Theorem 5.2.3. In order to determine the possible values of the deformation parameter t, we compute

$$\langle \omega^2, \sigma \wedge \bar{\sigma} \rangle = \frac{2nm|q_1|^2|q_2|^2}{(2\pi)^2(1 + |q_1|^2)^2(1 + |q_2|^2)^2},$$

whose maximum value is $\frac{nm}{8(2\pi)^2}$. Therefore, the family of GK structures exists for $|t| < \frac{8\pi}{\sqrt{nm}}$. ■

Example 5.2.8. Consider $\mathbb{P}^2$, with Fubini-Study form ω_{FS}. In a chart $\mathbb{C}^2$, with standard $\mathbb{T}^2$ action, the Kähler potential is given by

$$K(q_1, q_2) = \frac{1}{2\pi}\log(1 + |q_1|^2 + |q_2|^2).$$

One computes that

$$\langle \omega^2, \sigma \wedge \bar{\sigma} \rangle = \frac{2|q_1|^2|q_2|^2}{(2\pi)^2(1 + |q_1|^2 + |q_2|^2)^3},$$

which has maximum value $\frac{2}{27(2\pi)^2}$. Hence, we get a family of GK structures for $|t| < 6\sqrt{3}\pi$. ■

Chapter 6

The Picard group

In this chapter we focus on generalized Kähler structures where the two holomorphic Poisson structures are isomorphic. In this case, by choosing an isomorphism, we can assume that the two holomorphic Poisson structures coincide and then study the self-Morita equivalences of the given Poisson structure. This leads to the notion of the *Picard group* of a Poisson structure (X, σ), first introduced by Weinstein and Bursztyn in the smooth category in [15]. Recall from section 2.2.2 that integrable Poisson manifolds are the objects of a bicategory **sStack**, with 1-morphisms given by holomorphic symplectic Morita equivalences. Taking isomorphism classes of 1-morphisms, we obtain a category where all morphisms are invertible.

Definition 6.0.1. The *holomorphic Picard groupoid* $\mathcal{PG}$ is the category whose objects are integrable holomorphic Poisson manifolds and whose morphisms are isomorphism classes of holomorphic symplectic Morita equivalences. The *Picard group* of a holomorphic Poisson manifold (X, σ) is the automorphism group of (X, σ) in $\mathcal{PG}$:

$$\mathrm{Pic}(X, \sigma) = \mathsf{Hom}_{\mathcal{PG}}((X, \sigma), (X, \sigma))\,.$$

Since Morita equivalences equipped with bisections also compose, it is possible to upgrade the above groupoid so that the morphisms are holomorphic symplectic Morita equivalences with brane bisections.

Definition 6.0.2. The *Picard groupoid with branes* $\mathcal{PG}^{\mathcal{L}}$ is the category whose objects are integrable holomorphic Poisson manifolds and whose morphisms are isomorphism classes of holomorphic symplectic Morita equivalences equipped with brane bisections. The automorphism group of the object (X, σ) is denoted by

$$\mathrm{Pic}^{\mathcal{L}}(X, \sigma) = \mathsf{Hom}_{\mathcal{PG}^{\mathcal{L}}}((X, \sigma), (X, \sigma))\,.$$

Remark 6.0.3. Morphisms between two fixed Poisson manifolds $\mathsf{Hom}_{\mathcal{PG}^{\mathcal{L}}}((X_+, \sigma_+), (X_-, \sigma_-))$ correspond, by Theorem 4.2.2, to degenerate GK structures of symplectic type.

There is a natural forgetful functor from $\mathcal{PG}^{\mathcal{L}}$ to $\mathcal{PG}$ which drops the data of the brane bisection. This gives rise to a homomorphism

$$\mathrm{Pic}^{\mathcal{L}}(X, \sigma) \to \mathrm{Pic}(X, \sigma), \qquad [(Z, \Omega, \lambda)] \mapsto [(Z, \Omega)].$$

The kernel of this map consists of objects where the underlying Morita equivalence is trivial, i.e. isomorphic to the Weinstein groupoid $(\Sigma(X), \Omega)$. Therefore, the kernel is given by the image of the following natural homomorphism

$$\mathrm{Bis}^{\mathcal{L}}(\Sigma(X)) \to \mathrm{Pic}^{\mathcal{L}}(X,\sigma), \qquad \lambda \mapsto [(\Sigma(X),\Omega,\lambda)],$$

where $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X))$ is the group of brane bisections in $(\Sigma(X),\Omega)$. Note that here we are viewing a brane bisection as a map $\lambda : X \to Z$ such that $\pi_- \circ \lambda = id_X$, the map $\phi_\lambda := \pi_+ \circ \lambda$ is a diffeomorphism, and $\lambda^*\mathrm{Im}(\Omega) = 0$.

Using Theorem 4.2.2, we are able to give a concrete description of $\mathrm{Pic}^{\mathcal{L}}(X,\sigma)$, as follows. An object $[(Z,\Omega,\lambda)]$ of this group is a Morita self-equivalence with brane bisection of the holomorphic Poisson structure (X,σ). Let M denote the underlying smooth manifold of X, I its complex structure, and $Q = -4\mathrm{Im}(\sigma)$. By Theorem 4.2.2, this self-equivalence with brane corresponds to a degenerate GK structure of symplectic type, i.e. a solution (I_+, I_-, Q, F) of equations 3.8 and 3.9. Since we are viewing the brane as a section λ of π_- in this correspondence, we obtain $I_- = I$, $F = \lambda^*\Omega$, and $I_+ = (\phi_\lambda^{-1})_*(I)$, where $\phi_\lambda = \pi_+ \circ \lambda$. Therefore, given the holomorphic Poisson structure, the remaining data is encoded by the real 2-form F and the diffeomorphism ϕ_λ. Using equation 3.8, we express I_+ as $I^F := I + QF$. Then equation 3.9 can be replaced by equation 3.10:

$$FI + I^*F + FQF = 0,$$

and the relation between ϕ_λ, I_+ and I_- can be expressed as

$$(\phi_\lambda)_*(I^F) = I.$$

Based on these observations, we define the following subgroup of $\mathrm{Diff}_Q(M) \ltimes \Omega^{2,\mathrm{cl}}(M)$, the semi-direct product of the group of diffeomorphisms preserving Q with the group of real closed 2-forms:

$$\mathrm{Aut}_{\mathcal{C}}(I,\sigma) = \{(\phi, F) \in \mathrm{Diff}_Q(M) \times \Omega^{2,\mathrm{cl}}(M) \mid FI + I^*F + FQF = 0 \text{ and } \phi_*(I^F) = I\},$$

with multiplication defined as follows:

$$(\phi_1, F_1) * (\phi_2, F_2) = (\phi_1 \circ \phi_2, \phi_2^* F_1 + F_2). \tag{6.1}$$

This is the group of *Courant automorphisms of* (I,σ) when the holomorphic Poisson structure is viewed as a generalized complex structure, as in section 3.1.3. See [46] for a study of this group. An upshot of the present discussion is the existence of a map between $\mathrm{Pic}^{\mathcal{L}}(X,\sigma)$ and $\mathrm{Aut}_{\mathcal{C}}(I,\sigma)$, and a consequence of Theorem 4.2.2 is the fact that these two groups are isomorphic.

Corollary 6.0.4. *There is an isomorphism of groups*

$$\chi : \mathrm{Pic}^{\mathcal{L}}(X,\sigma) \to \mathrm{Aut}_{\mathcal{C}}(I,\sigma), \qquad [Z,\Omega,\lambda] \mapsto (\pi_+ \circ \lambda, \lambda^*\Omega).$$

Proof. The above discussion shows that the map χ is well-defined. Hence it remains to show that χ is a homomorphism and a bijection.

Step 1: χ is a group homormophism. Let (Z_1,Ω_1,λ_1) and (Z_2,Ω_2,λ_2) be Morita equivalences with brane bisections. The product of the bimodules is given by the quotient

$$Z_1 * Z_2 = (Z_1 \times_X Z_2)/\Sigma(X),$$

with symplectic form Ω defined via symplectic reduction, and the product of the bisections is given by

$$\lambda_1 * \lambda_2(x) = [\lambda_1 \circ \pi_{+,2} \circ \lambda_2(x), \lambda_2(x)].$$

Therefore $\pi_+ \circ (\lambda_1 * \lambda_2) = (\pi_{+,1} \circ \lambda_1) \circ (\pi_{+,2} \circ \lambda_2)$, and

$$\begin{aligned}(\lambda_1 * \lambda_2)^*\Omega &= (\lambda_1 \circ \pi_{+,2} \circ \lambda_2)^*\Omega_1 + \lambda_2^*\Omega_2 \\ &= (\pi_{+,2} \circ \lambda_2)^*(\lambda_1^*\Omega_1) + \lambda_2^*\Omega_2,\end{aligned}$$

which, according to equation 6.1, is the product of $(\pi_{+,1} \circ \lambda_1, \lambda_1^*\Omega_1)$ and $(\pi_{+,2} \circ \lambda_2, \lambda_2^*\Omega_2)$.

Step 2: χ is an isomorphism. We show this by defining an explicit inverse $\zeta : \mathrm{Aut}_{\mathcal{C}}(I, \sigma) \to \mathrm{Pic}^{\mathcal{L}}(X, \sigma)$. Given $(\phi, F) \in \mathrm{Aut}_{\mathcal{C}}(I, \sigma)$, Proposition 4.1.3 implies that we get the following element of the Picard

$$\begin{tikzcd} & (\Sigma(X), \Omega + t^*F) \arrow[dl, "\phi\circ t"'] \arrow[dr, "s"] & \\ (X, \sigma) & & (X, \sigma) \end{tikzcd}$$

where the brane bisection is given by the identity bisection ϵ. We therefore take this to define $\zeta(\phi, F)$. Theorem 4.2.2 then implies that ζ and χ are inverse to each other. □

As a result of this isomorphism, we immediately see that the kernel of the map $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X)) \to \mathrm{Pic}^{\mathcal{L}}(X, \sigma)$ is given by the subgroup $\mathrm{IsoLBis}(\Sigma(X))$ of holomorphic Lagrangian bisections of $(\Sigma(X), \Omega)$ which induce the identity diffeomorphism on M. Collecting these facts we get the following exact sequence of groups

$$1 \to \mathrm{IsoLBis}(\Sigma(X)) \to \mathrm{Bis}^{\mathcal{L}}(\Sigma(X)) \to \mathrm{Aut}_{\mathcal{C}}(I, \sigma) \to \mathrm{Pic}(X, \sigma). \tag{6.2}$$

We define the image of the last map to be the *real Picard group*, denoted $\mathrm{Pic}_{\mathbb{R}}(X, \sigma)$.

Remark 6.0.5. A central claim of this thesis has been that the holomorphic substructure underlying GK manifolds of symplectic type consists of holomorphic symplectic Morita equivalences, and that the additional real data needed to determine the metric consists of a brane bisection. This is reflected in the sequence 6.2 by the fact that $\mathrm{Aut}_{\mathcal{C}}(I, \sigma)$ is an extension of the real Picard group by the group of brane bisections.

Remark 6.0.6. Bursztyn and Fernandes studied a sequence similar to 6.2 in [13, Theorem 1.1] in the setting of real smooth Poisson geometry. The above may be seen as a generalization of their results to the case of holomorphic Poisson structures.

Note that the sequence 6.2 has the following exact subsequence

$$1 \to \mathrm{IsoLBis}(\Sigma(X)) \to \mathrm{LBis}(\Sigma(X)) \to \mathrm{Aut}(I, \sigma) \to \mathrm{Pic}(X, \sigma), \tag{6.3}$$

where $\mathrm{LBis}(\Sigma(X))$ is the group of holomorphic Lagrangian bisections, and $\mathrm{Aut}(I, \sigma)$ is the group of holomorphic Poisson isomorphisms of (I, σ). The group $\mathrm{Aut}(I, \sigma)$ sits inside $\mathrm{Aut}_{\mathcal{C}}(I, \sigma)$ as the subgroup of elements of the form $(\phi, 0)$. The image of the last map is known in [15] as the group of *Outer automorphisms*, denoted $\mathrm{OutAut}(\Sigma(X))$.

Remark 6.0.7. There is a version of the above exact sequence 6.3 in the setting of real smooth Poisson geometry. This was studied by Bursztyn and Weinstein in [15, Proposition 5.1] and then by Bursztyn and Fernandes in [13, Corollary 3.6].

Trivial Poisson structure

The Picard group of the zero Poisson structure on a smooth manifold was computed in [15]. We derive here the analogue in the holomorphic category.

Proposition 6.0.8. *Let (X, I) be a complex manifold, equipped with the zero Poisson structure. Then its Picard group is given by*

$$\mathrm{Pic}(X, 0) = \mathrm{Diff}(X, I) \ltimes H^1(\Omega_X^{1,cl}),$$

where $H^1(\Omega_X^{1,cl})$ is the first cohomology group of the sheaf of closed holomorphic 1-forms, which can be expressed as follows

$$H^1(\Omega_X^{1,cl}) = \frac{(\Omega^{(1,1)} \oplus \Omega^{(2,0)}) \cap ker(d)}{d(\Omega^{(1,0)})}.$$

Furthermore, the real Picard group is given by the elements where the cohomology class has a real representative, namely

$$\mathrm{Pic}_{\mathbb{R}}(X, 0) = \mathrm{Diff}(X, I) \ltimes \frac{\Omega^{(1,1)}(X, \mathbb{R}) \cap ker(d)}{d\Omega^{(1,0)}}.$$

Proof. Let (Z, Ω) be a holomorphic symplectic Morita equivalence, with $\pi_\pm$ denoting the two projection maps. The fibres of $\pi_\pm$ are Lagrangian, and therefore, they must coincide since they are also symplectic orthogonal to each other. There is a well-defined bijection $\phi : X \to X$, defined by sending a point x to $\pi_+(\pi_-^{-1}(x))$, and we can check that it is holomorphic by using local holomorphic Lagrangian bisections to represent it locally. It then follows that the map π_+ factors as $\phi \circ \pi_-$, and the left and right actions of T^*X are related by ϕ, in the sense that for $z \in \pi_-^{-1}(x)$, and $\alpha_x \in T_x^*X$, we have

$$\phi_*(\alpha_x) + z = z + \alpha_x,$$

where we are using $+$ to denote the groupoid action. The Morita equivalence Z therefore consists of the data of a symplectic affine bundle for T^*X, and an isomorphism ϕ of X. Isomorphisms of Morita equivalences do not change the isomorphism of X, and are just given by isomorphisms of affine bundles. Affine bundles for T^*X are classified by the sheaf of holomorphic 1-forms, and in order for the bundle to respect the symplectic structure, these forms must be closed. Therefore, we see that the Picard group is given by

$$\mathrm{Pic}(X, 0) = \mathrm{Diff}(X, I) \times H^1(\Omega_X^{1,cl}).$$

The Dolbeault representation of an element of the Picard comes from choosing a global smooth bisection, and the group structure can be determined by studying the composition of Morita equivalences, as in the proof of Corollary 6.1. $\square$

If we are on a compact Kähler manifold, then $\mathrm{Pic}(X, 0) = \mathrm{Diff}(X, I) \ltimes (H^{(1,1)} \oplus H^{(2,0)})$, with the real Picard group given by $\mathrm{Pic}_{\mathbb{R}}(X, 0) = \mathrm{Diff}(X, I) \ltimes H^{(1,1)}(\mathbb{R})$. This suggests a definition of the *generalized*

Kähler cone of a holomorphic Poisson structure. First, the analogue of $H^{(1,1)}(\mathbb{R})$ is given by the right coset space

$$\mathrm{Pic}_{\mathbb{R}}(X,\sigma)/\mathrm{OutAut}(\Sigma(X)).$$

Note that the set of degenerate GK structures of symplectic type, which arise from self-Morita equivalences with brane bisection, corresponds to the right coset space

$$\mathrm{Aut}_{\mathcal{C}}(I,\sigma)/\mathrm{Aut}(I,\sigma).$$

We then define the generalized Kähler cone to be the subset of $\mathrm{Pic}_{\mathbb{R}}(X,\sigma)/\mathrm{OutAut}(\Sigma(X))$ consisting of elements which admit a positive representative.

6.1 The Picard algebra and integration

We now turn to the infinitesimal versions of the groups considered above, and the corresponding exponential maps. The Lie algebra of $\mathrm{Aut}_{\mathcal{C}}(I,\sigma)$ is the following subalgebra of $\mathfrak{X}_Q(M) \ltimes \Omega^{2,cl}(M)$, the semi-direct product of the Lie algebra of vector fields preserving Q, with the vector space of closed 2-forms:

$$\mathfrak{aut}_{\mathcal{C}}(I,\sigma) = \{(V,\omega) \in \mathfrak{X}_Q(M) \times \Omega^{2,cl}(M) \mid \omega I + I^*\omega = 0 \text{ and } \mathcal{L}_V I = Q\omega\}.$$

This is the Lie algebra of *infinitesimal Courant symmetries* of the holomorphic Poisson structure (I,σ). The bracket is given by

$$[(V,\omega),(W,\eta)] = ([V,W], \mathcal{L}_V\eta - \mathcal{L}_W\omega).$$

Let $(V,\omega) \in \mathfrak{aut}_{\mathcal{C}}(I,\sigma)$ be an element of this Lie algebra. This defines an infinitesimal automorphism of the Courant algebroid $T \oplus T^*$ and therefore it integrates to the following family of automorphisms of $T \oplus T^*$:

$$(\phi_t, F_t = \int_0^t (\phi_s^*\omega)ds),$$

where ϕ_t is the flow of the vector field V. In [46, Section 7] it is shown that this family lies in $\mathrm{Aut}_{\mathcal{C}}(I,\sigma)$ for all t (where it is defined), and therefore this family defines the 1-parameter subgroup integrating (V,ω). Hence we have the following result.

Lemma 6.1.1. *The exponential map*

$$\exp : \mathfrak{aut}_{\mathcal{C}}(I,\sigma) \to \mathrm{Aut}_{\mathcal{C}}(I,\sigma) \cong \mathrm{Pic}^{\mathcal{L}}(X,\sigma)$$

is given by flowing the above family of automorphisms to $t = 1$:

$$(V,\omega) \mapsto (\phi_1, F_1 = \int_0^1 (\phi_s^*\omega)ds).$$

Now consider the group $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X))$ of brane bisections. Since these are really just the Lagrangian bisections for the groupoid integrating Q, we know from Proposition 2.2.13 that its Lie algebra is given by the space of real closed 1-forms $\Omega^{1,cl}(M)$, with Lie bracket induced by Q:

$$[\alpha,\beta] = dQ(\alpha,\beta).$$

Let's recall the description of the exponential map $\exp : \Omega^{1,cl}(M) \to \mathrm{Bis}^{\mathcal{L}}(\Sigma(X))$ from Section 2.2.1. Given $\alpha \in \Omega^{1,cl}(M)$, we can consider the vector field induced by $t^*\alpha$ on the Weinstein groupoid with respect to the imaginary part of the symplectic form $\omega = \mathrm{Im}(\Omega)$:

$$V_{t^*\alpha} = \omega^{-1}(t^*\alpha).$$

This is the right-invariant vector field associated to α when it is viewed as a section of the Lie algebroid $T^*_Q M$ of $(\Sigma(X), \omega)$. It is t-related to the vector field $Q(\alpha)$ on X. Let ϕ_t be the flow of $V_{t^*\alpha}$. Then the 1-parameter subgroup integrating α is given by the family $\lambda_t = \phi_t \circ \epsilon$ of bisections, where ϵ is the identity bisection, so that

$$\exp(\alpha) = \lambda_1.$$

Mapping the 1-parameter subgroup λ_t to $\mathrm{Aut}_{\mathcal{C}}(I, \sigma)$ and taking the derivative at $t = 0$ determines the induced Lie algebra morphism $\Omega^{1,cl}(M) \to \mathfrak{aut}_{\mathcal{C}}(I, \sigma)$: $\lambda_t \in \mathrm{Bis}^{\mathcal{L}}(\Sigma(X))$ gets sent to the family $[\Sigma(X), \Omega, \lambda_t] \in \mathrm{Pic}^{\mathcal{L}}(X, \sigma)$ of Morita equivalences with brane bisections, and this corresponds in $\mathrm{Aut}_{\mathcal{C}}(I, \sigma)$ to the family of Courant automorphisms given by $(\psi_t, \int_0^t (\psi_s^* d^c\alpha) ds)$, where ψ_t is the flow of $Q(\alpha)$ and $d^c = i(\bar{\partial} - \partial)$. Therefore the map of Lie algebras is given by

$$\Omega^{1,cl}(M) \to \mathfrak{aut}_{\mathcal{C}}(I, \sigma), \qquad \alpha \mapsto (Q(\alpha), d^c\alpha).$$

One upshot of the present discussion is that the exponential map for $\mathrm{Aut}_{\mathcal{C}}(I, \sigma)$ has a particularly nice description when it is applied to elements in the image of $\Omega^{1,cl}(M)$.

Proposition 6.1.2. *Let $\alpha \in \Omega^{1,cl}(M)$ be a real closed 1-form and let $(Q(\alpha), d^c\alpha) \in \mathfrak{aut}_{\mathcal{C}}(I, \sigma)$ be the infinitesimal Courant symmetry that it determines. Exponentiating this symmetry yields a family of holomorphic symplectic Morita equivalences with brane bisections. This family has the following simple form:*

$$[(\Sigma(X), \Omega, \phi_t \circ \epsilon)],$$

where $(\Sigma(X), \Omega)$ is the Weinstein groupoid viewed as a trivial Morita equivalence, ϕ_t is the flow of the vector field $(\mathrm{Im}\Omega)^{-1}(t^*\alpha)$ on the groupoid, and $\phi_t \circ \epsilon$ is the result of applying this flow to the identity bisection.

Another upshot is the infinitesimal version of the sequence 6.2:

$$0 \to \Omega^{(1,0),cl}_{bas}(X) \to \Omega^{1,cl}(M) \to \mathfrak{aut}_{\mathcal{C}}(I, \sigma) \to \mathfrak{pic}_{\mathbb{R}}(I, \sigma) \to 0,$$

which is a generalization of [13, Theorem 1.2] to the setting of holomorphic Poisson structures. In this sequence

$$\Omega^{(1,0),cl}_{bas}(X) = \{\eta \in \Omega^{(1,0)}(X) \mid d\eta = 0,\ \sigma(\eta) = 0\},$$

the vector space of closed holomorphic 1-forms which are in the kernel of σ, and hence are *basic*, meaning that they descend to the space of symplectic leaves. This maps injectively to $\Omega^{1,cl}(M)$ by taking the real part. The space $\mathfrak{pic}_{\mathbb{R}}(I, \sigma)$ is the *real Picard algebra*, and is defined so that the above sequence is exact. We view it as the Lie algebra of the real Picard group.

Chapter 7

Hamiltonian flows

In this chapter we give two applications of the ideas of Chapter 6. First we explain how to deform GK structures using the flow of Hamiltonian vector fields, and then we show that locally, all GK structures arise in this way by deforming away from a standard degenerate GK structure.

7.1 Generalized Kähler metrics via Hamiltonian flows

In this section we apply the ideas of Chapter 6 to give a method of construction and deformation of generalized Kähler metrics. The basic idea is as follows: Theorem 4.2.2 allows us to view GK structures of symplectic type as Morita equivalences with brane bisection, which are morphisms in the groupoid $\mathcal{PG}^{\mathcal{L}}$. So, it is possible to compose them. More precisely, if $(Z_1, \Omega_1, \mathcal{L}_1)$ and $(Z_2, \Omega_2, \mathcal{L}_2)$ are degenerate GK structures of symplectic type going between holomorphic Poisson structures $(X_1, \sigma_1), (X_2, \sigma_2)$ and $(X_2, \sigma_2), (X_3, \sigma_3)$ respectively, then we may compose them to get a degenerate GK structure going between (X_1, σ_1) and (X_3, σ_3). In particular, there is an action of the group $\mathrm{Pic}^{\mathcal{L}}(X_+, \sigma_+)$ on the space of morphisms $\mathcal{PG}^{\mathcal{L}}((X_+, \sigma_+), (X_-, \sigma_-))$ and we can use this to *deform* GK structures of symplectic type. Indeed, all of the constructions of GK metrics contained in [57, 56, 46] are special cases of this.

So let $(Z, \Omega, \mathcal{L}) \in \mathcal{PG}^{\mathcal{L}}((X_+, \sigma_+), (X_-, \sigma_-))$ be a GK structure viewed as a Morita equivalence with brane bisection

$$\begin{tikzcd}[column sep=small] & (Z, \Omega, \mathcal{L}) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ (X_+, \sigma_+) & & (X_-, \sigma_-) \end{tikzcd}$$

and let $\zeta(\phi, F)$ be an element of $\mathrm{Pic}^{\mathcal{L}}(X_+, \sigma_+)$ corresponding to the Courant automorphism $(\phi, F) \in \mathrm{Aut}_{\mathcal{C}}(I_+, \sigma_+)$. Composing with the above Morita equivalence, we obtain $\zeta(\phi, F) * (Z, \Omega, \mathcal{L}) = (Z, \Omega + \pi_+^* F, \mathcal{L})$, again a morphism in $\mathcal{PG}^{\mathcal{L}}((X_+, \sigma_+), (X_-, \sigma_-))$ as depicted below.

$$\begin{tikzcd}[column sep=small] & (Z, \Omega + \pi_+^* F, \mathcal{L}) \arrow[dl, "\phi \circ \pi_+"'] \arrow[dr, "\pi_-"] & \\ (X_+, \sigma_+) & & (X_-, \sigma_-) \end{tikzcd}$$

Since the positivity of the induced metric is an open condition, this will define a new GK structure if

(ϕ, F) is 'close enough' to the identity.

Now the idea of the construction in [46] is to obtain the element $\zeta(\phi, F) \in \mathrm{Pic}^{\mathcal{L}}(X_+, \sigma_+)$ by integrating an infinitesimal symmetry via the exponential map of Lemma 6.1.1. So given $(V, \omega) \in \mathfrak{aut}_{\mathcal{C}}(I_+, \sigma_+)$, we exponentiate it to the family $\zeta(\exp(t(V, \omega)))$ in $\mathrm{Pic}^{\mathcal{L}}(X_+, \sigma_+)$. Then $\zeta(\exp(t(V, \omega)) * (Z, \Omega, \mathcal{L})$ defines a family of degenerate GK structures deforming the initial structure, and the metric will remain positive-definite for sufficiently small t.

Proposition 6.1.2 explains that exponentiating the infinitesimal symmetries in the image of $\Omega^{1,cl}(M) \to \mathfrak{aut}_{\mathcal{C}}(I_+, \sigma_+)$ simply amounts to applying a flow to the identity bisection. The GK deformations that arise from these symmetries are similarly simple.

Proposition 7.1.1. *Let $(Z, \Omega, \mathcal{L})$ be a GK structure of symplectic type, viewed as a holomorphic symplectic Morita equivalence with brane bisection going between the holomorphic Poisson structures $(I_\pm, \sigma_\pm)$. Let $\alpha \in \Omega^{1,cl}(M)$ be a real closed 1-form, and let $(Q(\alpha), d^c\alpha) \in \mathfrak{aut}_{\mathcal{C}}(I_+, \sigma_+)$ be the infinitesimal Courant symmetry that it determines. Exponentiating this symmetry, we get a 1-parameter family of Courant automorphisms which act on $(Z, \Omega, \mathcal{L})$ to produce a family of degenerate GK structures of symplectic type deforming the given one. This family has the following simple form:*

$$[(Z, \Omega, \mathcal{L}_t = \eta_t(\mathcal{L}))],$$

where η_t is the flow of the vector field $V_{\pi_+^*\alpha} = (\mathrm{Im}\Omega)^{-1}(\pi_+^*\alpha)$, and $\mathcal{L}_t = \eta_t(\mathcal{L})$ is the result of applying this flow to the brane bisection $\mathcal{L}$.

Proof. Let $\lambda_t = \phi_t \circ \epsilon$ be the 1-parameter subgroup corresponding to α in $\mathrm{Bis}^{\mathcal{L}}(\Sigma(X_+))$, where ϕ_t is the flow of $V_{t^*\alpha} = \omega^{-1}(t^*\alpha)$ on $\Sigma(X_+)$. By Proposition 6.1.2, the family of degenerate GK structures obtained by exponentiating $(Q(\alpha), d^c\alpha)$ and acting on the given GK structure is given by the composition of $(\Sigma(X_+), \Omega_+, \lambda_t)$ and $(Z, \Omega, \mathcal{L})$. This gives $(Z, \Omega, \mathcal{L}_t = \lambda_t * \mathcal{L})$, where

$$(\lambda_t * \mathcal{L})(x) = \lambda_t(\pi_+ \circ \mathcal{L}(x)) * \mathcal{L}(x),$$

and where we are using the action $*$ of $\Sigma(X_+)$ on Z. Note that we are abusing notation by using $\mathcal{L}$ to denote both the section of π_- and its image in Z. Now let $V_{\pi_+^*\alpha} = \mathrm{Im}(\Omega)^{-1}(\pi_+^*\alpha)$, a vector field on Z, and let η_t be its flow. We claim that $\mathcal{L}_t = \eta_t(\mathcal{L})$. This can be seen as follows. Using the brane $\mathcal{L}$ we define a smooth symplectomorphism

$$\Phi : (\Sigma(X_+), \mathrm{Im}(\Omega_+)) \to (Z, \mathrm{Im}(\Omega)), \qquad g \mapsto g * (r \circ s)(g),$$

where $r : X_+ \to Z$ is the section of π_+ induced by $\mathcal{L}$, and s is the source map of the Weinstein groupoid. This map satisfies the relation $\pi_+ \circ \Phi = t$, and therefore $d\Phi(V_{t^*\alpha}) = V_{\pi_+^*\alpha}$. This implies that Φ intertwines the flows ϕ_t and η_t of the vector fields $V_{t^*\alpha}$ and $V_{\pi_+^*\alpha}$, respectively. Then, by the definitions of Φ and $\mathcal{L}_t$ we see that on the one-hand

$$\Phi \circ \lambda_t \circ \pi_+ \circ \mathcal{L}(x) = \lambda_t(\pi_+ \circ \mathcal{L}(x)) * \mathcal{L}(x) = \mathcal{L}_t(x),$$

and by the property that Φ intertwines ϕ_t and η_t we see that on the other hand

$$\Phi \circ \lambda_t \circ \pi_+ \circ \mathcal{L}(x) = \eta_t \circ \Phi(1_{\pi_+ \circ \mathcal{L}(x)}) = \eta_t(\mathcal{L}(x)).$$

□

Remark 7.1.2. In Propositions 6.1.2 and 7.1.1, we may specialize to the case of exact 1-forms $\alpha = dK$, in which case the families we obtain are given by flowing the brane bisections by Hamiltonian vector fields.

Example 7.1.3. Recall from Example 4.3.3 that the Morita equivalence of a hyper-Kähler structure (M, I, J, K, g) is given by

$$(Z, \Omega) = (X_+, \Omega_+) \times (X_-, -\Omega_-),$$

with brane bisection $\mathcal{L}$ given by the diagonally embedded copy of M. It was observed in [2] that such a GK structure may be deformed to a new GK structure which is not hyper-Kähler using a real-valued function f. In the present setting we can view this as a deformation obtained by flowing the brane bisection using a Hamiltonian vector field of f. More precisely, given a real valued-function f, Proposition 7.1.1 says that the brane is flowed by the Hamiltonian vector field of $\pi_+^* f$, using the imaginary part of the symplectic form on Z, namely, $\mathrm{Im}(\Omega_+, -\Omega_-) = (\omega_K, -\omega_K)$. Therefore, the brane is given by the flow of

$$(\omega_K^{-1}(df), 0),$$

and so if ϕ_t is the flow of the Hamiltonian vector field of f with respect to ω_K, then the deformed brane is given by

$$\mathcal{L}_t = \{(\phi_t(m), m) \mid m \in M\},$$

which is the graph of ϕ_t. ■

7.2 Universal local construction via time-dependent flows

Let (M, I, σ) be a holomorphic Poisson structure, which determines a degenerate GK structure as in Example 3.1.9. As explained in Example 4.3.1, the corresponding Morita equivalence provided by Theorem 4.2.2 is the symplectic groupoid $(\Sigma(X), \Omega, \epsilon)$ integrating σ, with the identity bisection ϵ as its brane bisection. Propositions 6.1.2 and 7.1.1 tell us that given a real-valued function f we can construct a family of degenerate GK structures deforming the given one by deforming the identity bisection ϵ using the Hamiltonian vector field of $t^* f$ on the groupoid: $V_{t^* df} = \omega^{-1}(dt^* f)$, where here we decompose $\Omega = B + i\omega$ into its real and imaginary parts. In this section, we show that, locally, all degenerate GK structures of symplectic type arise in this way if we allow the function f to depend on time. This gives a universal local construction (similar to the one using a generalized Kähler potential) which places a greater emphasis on the underlying holomorphic Poisson geometry. In effect, it tells us that locally a GK structure of symplectic type is determined by a holomorphic Poisson structure and a single time-dependent real-valued function.

Let $(Z, \Omega, \mathcal{L})$ be a GK structure viewed as a Morita equivalence with brane bisection, and let $z \in \mathcal{L}$ be a point on the brane. Choose a local holomorphic Lagrangian bisection Λ passing through the point z and let $U := \pi_-(\Lambda) \subseteq X_-$, and $V := \pi_+(\Lambda) \subseteq X_+$. Note first that this bisection induces a holomorphic Poisson isomorphism $\phi : (U, \sigma_-) \to (V, \sigma_+)$; this means that the two holomorphic Poisson structures underlying a GK structure of symplectic type are always locally isomorphic, although not in a canonical way. Now consider the local Morita equivalence $\pi_+^{-1}(V) \cap \pi_-^{-1}(U)$ going between (U, σ_-) and (V, σ_+). Using the Lagrangian Λ, we can identify this Morita equivalence with the trivial Morita equivalence

$t^{-1}(U) \cap s^{-1}(U) \subseteq \Sigma(X_-)$. That is to say, we have a holomorphic symplectomorphism

$$\Phi : t^{-1}(U) \cap s^{-1}(U) \to \pi_+^{-1}(V) \cap \pi_-^{-1}(U), \qquad g \mapsto \Lambda(t(g)) * g, \tag{7.1}$$

satisfying $\pi_+ \circ \Phi = \phi \circ t$, as well as $\pi_- \circ \Phi = s$ and $\Phi \circ \epsilon = \Lambda$. We view this as a *groupoid chart*, in the sense that it identifies a neighbourhood of z in Z with a neighbourhood of the zero section in $\Sigma(X_-)$. The chart is adapted to the groupoid structure, unlike the Darboux charts considered in Chapter 5. In this chart, the brane $\mathcal{L}$ intersects the identity bisection at the point z. Our goal is to describe $\mathcal{L}$ as a Hamiltonian flow applied to the zero section: for this, we require a family of brane bisections $\mathcal{L}_t$ interpolating between $\mathcal{L}$ and Λ. First we show that such a family exists.

Proposition 7.2.1. *There is a local family of brane bisections interpolating between a given brane $\mathcal{L}$ and a holomorphic Lagrangian bisection Λ.*

Such a family of brane bisections consists of a family of Lagrangian submanifolds for $\omega = \mathrm{Im}(\Omega)$ which is transverse to both $K_\pm = \ker(d\pi_\pm)$ at all times. Note that since K_+ and K_- are symplectic orthogonal, a Lagrangian L which is transverse to K_+ is automatically transverse to K_-:

$$L \cap K_- = L^\omega \cap K_+^\omega = (L + K_+)^\omega = TZ^\omega = 0.$$

The linear version of this problem has an immediate solution: let (V, ω) be a $2n$-dimensional (real) symplectic vector space and let K be an arbitrary n-dimensional subspace. Let $M_{V,K}$ denote the space of Lagrangians in V which are transverse to K; this is a connected open subset of the Lagrangian Grassmannian, showing that the linear version of such an interpolation is available.

Proof of Proposition 7.2.1. Choose a holomorphic Darboux chart centred at the point $z : (TZ_z, \Omega_z) \cong (Z, \Omega)$, and let Λ be a holomorphic Lagrangian subspace of TZ_z which is transverse to $K_\pm$. This defines a (local) holomorphic Lagrangian bisection Λ. The tangent space $L = T_z\mathcal{L}$ defines a Lagrangian subspace of (TZ_z, ω_z) which is also transverse to $K_\pm$. Since M_{TZ_z,K_+} is connected, we choose a family L_t of Lagrangian subspaces of (TZ_z, ω_z) which remain transverse to $K_\pm$ for all time interpolating between Λ and L. We can view this as a family of brane bisections going from Λ to the brane L. Hence it remains to find a path going from L to $\mathcal{L}$. For this choose a Weinstein neighbourhood of L: $(T^*L, \Omega_0) \cong (Z, \omega)$. In this chart, $\mathcal{L}$ is given by the graph of a closed 1-form $\alpha \in \Omega^1(L)$. Then $\mathcal{L}_t = Gr(t\alpha)$ defines a family of Lagrangians interpolating between L and $\mathcal{L}$. Since $T\mathcal{L}_z = TL_z$ it follows that $T(\mathcal{L}_t)_z = TL_z$ for all t implying that this family is transverse to $K_\pm$ at all times. Combining the two families, we obtain a family of branes interpolating between the holomorphic Lagrangian Λ and the brane bisection $\mathcal{L}$. This family fixes the point z, and at all times the tangent space at z is transverse to $K_\pm$. Thus we obtain the required interpolation on a (possibly smaller) neighbourhood of z. $\square$

Having the interpolating family of branes, we wish to describe it as a Hamiltonian flow. For this, we return to the groupoid chart $t^{-1}(U) \cap s^{-1}(U)$ where we have the following data:

1. A holomorphic symplectic groupoid $(\Sigma(U), \Omega_-)$ integrating (U, σ_-), with underlying imaginary part the smooth real symplectic groupoid $(\Sigma(U), \omega)$;

2. Over a fixed neighbourhood of z, $W \subseteq U$, we have a family of Lagrangian bisections of $(\Sigma(U), \omega)$,

viewed as sections of the source s,

$$\lambda_t : W \to (\Sigma(U), \omega),$$

such that λ_0 is the identity bisection and λ_1 is the given brane bisection $\mathcal{L}$ viewed in the groupoid chart. Note that $\lambda_t(z) = z$ for all time t.

Now let $\psi_t := t \circ \lambda_t$ denote the resulting family of Poisson diffeomorphisms (for the Poisson structure Q), and let $W_t = \psi_t(W)$. Left multiplication by the bisection defines the following family of symplectomorphisms:

$$\tau_t : t^{-1}(W) \to t^{-1}(W_t), \; g \mapsto \lambda_t(t(g)) * g,$$

which satisfy $s \circ \tau_t = s$ and $t \circ \tau_t = \psi_t \circ t$, as well as $\tau_t \circ \epsilon = \lambda_t$. This family of symplectomorphisms defines a time-dependent vector field $Y_t \in \mathcal{X}^1(t^{-1}(W_t))$ via the equation

$$Y_t(\tau_t(g)) = \frac{d}{dt}\tau_t(g).$$

As explained in [113] this family of vector fields is Hamiltonian for a t-basic closed 1-form:

$$\iota_{Y_t}\omega = t^*(\alpha_t),$$

where $\alpha_t \in \Omega^1(W_t)$ is a closed time-dependent 1-form. In fact, the restriction of Y_t to the bisection gives the following vector field along λ_t:

$$X_t(x) := Y_t(\lambda_t(x)) = \frac{d}{dt}\lambda_t(x).$$

Since λ_t is a section of s, $X_t \in \ker(ds)$ and hence $\omega(X_t)$ is in the image of t^*, defining the form α_t. Note that since t is a Poisson map we have

$$t_*(Y_t) = Q(\alpha_t),$$

and so ψ_t is the flow of this Hamiltonian vector field. The time-dependent form $\alpha_t : [0,1] \to \Omega^1_{cl}$ is the infinitesimal version of the family of branes $\lambda_t : [0,1] \to \mathrm{Bis}^{\mathcal{L}}(\Sigma)$.

Now choose a neighbourhood W' of z such that $W' \subseteq W_t$ for all time t. Restricting to this neighbourhood, we have $\alpha_t \in \Omega^{1,cl}(W')$, and if we assume that W' is contractible then α_t are exact. Choose a primitive: let $f_t \in C^\infty(W')$ be a time-dependent function such that $df_t = \alpha_t$. We conclude that it is now possible to describe the GK structure purely in terms of this function and the holomorphic Poisson structure (I_-, σ_-):

Theorem 7.2.2. *Let $(Z, \Omega, \mathcal{L})$ be a Generalized Kähler structure of symplectic type, viewed as a holomorphic symplectic Morita equivalence with brane bisection going between holomorphic Poisson structures $(X_\pm, \sigma_\pm)$ with common imaginary part $-\frac{1}{4}Q$. Let $z \in \mathcal{L}$ be a chosen point on the brane. Then*

1. *It is possible to choose a family of local brane bisections $\mathcal{L}_t$ such that $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_0 = \Lambda$, a holomorphic Lagrangian bisection, and such that $z \in \mathcal{L}_t$ for all $t \in [0,1]$.*

2. *There is a (locally defined) time-dependent real-valued function $f_t \in C^\infty(X_-, \mathbb{R})$ such that in a neighbourhood of z the family of brane bisections $\mathcal{L}_t$ is given by $\tilde{\tau}_t(\Lambda)$, where $\tilde{\tau}_t$ is the flow of the Hamiltonian vector field of $(\phi^{-1}\pi_+)^* f_t$ with respect to the symplectic form* $\mathrm{Im}(\Omega)$.

3. *In a neighbourhood of $\pi_-(z)$ the GK structure is given by the data (I_+, I_-, Q, F), where I_- is the given complex structure on X_-, $I_+ = \psi_1^*(I_-)$ and*

$$F = \int_0^1 \psi_t^*(d^c_- df_t)dt,$$

for ψ_t the flow of the Hamiltonian vector field $X_{f_t} = Q(df_t)$, and where $d^c_- = i(\bar\partial_{I_-} - \partial_{I_-})$.

In other words, in a neighbourhood of any point, a GK structure of symplectic type is determined by a holomorphic Poisson structure, together with a time-dependent real-valued function, via the Hamiltonian flow construction of Section 7.1.

Proof. It remains only to prove the formula for $F = \lambda_1^*\Omega_-$. Differentiating the pullback $\lambda_t^*\Omega_-$ gives

$$\frac{d}{dt}(\lambda_t^*\Omega_-) = \lambda_t^*\mathcal{L}_{Y_t}\Omega_- = \lambda_t^* dI^*\omega(Y_t) = \lambda_t^* dI^* t^*\alpha_t = \psi_t^* dI_-^*\alpha_t = \psi_t^* d^c_-\alpha_t = \psi_t^*(d^c_- df_t).$$

Therefore $\lambda_1^*\Omega_- = \int_0^1 \frac{d}{dt}(\lambda_t^*\Omega_-)dt = \int_0^1 \psi_t^*(d^c_- df_t)dt.$ □

Example 7.2.3. Example 5.1.1 of a Kähler structure showed how the generalized Kähler potential of Chapter 5 generalizes the usual Kähler potential. However, another look at this example shows that what we were doing is actually an instance of what is described in the present section. Namely, given a holomorphic Lagrangian L, viewed as a section λ, we used the groupoid structure to induce a local isomorphism of Morita equivalences:

$$(T^*X, \Omega_0) \to (Z, \Omega), \qquad \alpha_x \mapsto \alpha_x + \lambda(x),$$

so that our Darboux chart in this case is actually a groupoid chart in the sense of (7.1). Now we view the brane bisection $\mathcal{L}$ in this chart as the graph of a $(1,0)$-form $\eta = I^*\alpha + i\alpha$, where α is a closed 1-form. For our interpolating family of branes, we may choose $\mathcal{L}_t = Gr(t\eta)$. Then the induced family of closed 1-forms on X is given by the time-independent form α. As in example 5.1.1, we choose a real-valued function K on X such that $\alpha = -\frac{1}{2}dK$. Then noting that the Hamiltonian vector field $X_K = 0$, and appealing to Theorem 7.2.2, we see that the Kähler form is given by

$$\omega = -\frac{1}{2}d^c dK = i\partial\bar\partial K.$$

Therefore the construction of Theorem 7.2.2 provides an alternate generalization of the Kähler potential which is more closely adapted to the underlying groupoid structure. ■

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